Networked Consensus Agents Dynamics through Matrix Eigenvalue

Toshihiko Yamamoto and Akira Namatame

Department of Computer Science, National Defense Academy of Japan,
Yokosuka, Hashirimizu 1-10-20, Japan
{ g47038, nama } @nda.ac.jp

Abstract. The collective behavior system depends strongly on the efficiency of communication paths, in other words, network topology. Network connectivity affects the properties of networked agents. Synchronous behavior is also affected by the network structure. A consensus problem is also related to the problem of synchronization. A consensus algorithm is an interaction rule that specifies the information exchange between an agent and all of its neighbors on the network. A class of local control laws is investigated for a collection of agents that result in global order their motion in convergence of their movement to a common. In large groups of agents the information sharing should be local in some sense, due to energy limitation, reliability, and other constraints. However, local information exchange limits the speed of convergence of such protocols. A consensus protocol is an iteractive method that provides the group with a common coordination variable. In this paper, we describe the occurrence of flocking behavior and the underlying connection between agents. The occurrence of flocking behavior in the synchronization of networked agents is directly associated with the connectivity properties of the interconnection network. Each node of the network represents a dynamical system. Individual systems are coupled according to the network topology. It is possible to think about the network topology through local control action by exploiting the graph theoretic properties of the agent networks. We consider the network by the Laplacian matrix eigenvalue viewpoint.

Keywords: Consensus,Laplacian Matrix Eigenvalue,Flocking Behavior,Complex Network

1 Introduction

Flocking is this phenomenon that individuals all move with approximately the same motion, so that they remain together as a group. In nature, aggregations of large numbers of mobile organisms are also faced with the problem of organizing themselves efficiently. This selective pressure has led to the evolution of behavior such as flocking of birds, swarming of insects, herding of land animals and schooling of fish. Reynolds developed a method that creates realistic simulations of bird flocking[1]. This model is known as Boids Model. and, there are three interaction rules: 1) attraction (cohesion rule), 2) collision avoidance (separation rule), and 3) velocity matching (alignment rule) between the boids located within a certain radius. these 3 local rules create produce realistic flocking behavior.

In boids model, each agent has direct access to the whole scene 's geometric description, but flocking requires that it react only to flock mates within a certain small neighborhood around itself. The neighborhood is characterized by a distance and an angle, measured from the agent 's direction of flight.

We are interested in self-organization and group behavior about independent and autonomous agent. They are decentralized and doesn't share information. It is recognized that most complex systems in nature are organized as intricate network patterns [2][3]. Furthermore, Autonomous agents of conventional model need connected network structure as a prerequisite for the emergence of flocking behavior. On the other hand, Autonomous agents of our model doesn't need connected network structure and emerge flocking behavior. We think about the network topology by exploiting the graph theoretic properties of the agent networks. Exploiting modern results from algebraic graph theory, these properties are directly related to the topology of the network through the eigenvalues of the Laplacian of the graph.

2 Emergence Flocking Behavior

The flocking model consists of three simple steering behaviors which describe how an individual agent maneuvers based on the positions and velocities its nearby flock mates: Boids model is as follows[1] :

(1) Cohesion: steer to move towards the average position of the neighboring flockmates
(2) Separation: steer to avoid crowding the neighboring flockmates
(3) Alignment: steer towards the average heading of the neighboring flockmates

A flock often consists of multiple local interactions of each agent.agents. The cohesion rule causes each active flock member to try to orient its velocity vector in the direction of the centroid of the local flock. The degree of locality of the rule is determined by the sensor range of the active flock member, represented by the light colored circle. Separation behavior gives an agent the ability to maintain a certain separation distance from others nearby. This can be used to prevent agents from crowding together. To compute steering for separation, a search is first made to find other agents within the specified neighborhood. Alignment behavior gives an agent the ability to align itself with, that is, head in the same direction or speed as, other nearby agents. Steering for alignment can be computed by finding all agents in the local neighborhood.

A method is presented for flocking behavior of creatures, birds, fishes and so on, that can form herds by evading obstacles in airspace, terrain or ocean floor topography in 3D space while being efficient enough to run in real-time. This method involves making modifications to the boids model [1] as following.

The flocking algorithm works as follows: For a given agent, centroids are calculated using the sensor characteristics associated with each flocking rule. Next, the velocity vector the given agent should follow to carry out the rule is calculated for each of the rules. These velocity vectors are then weighted according to the rule strength and added together to give an overall velocity vector demand. Finally, this velocity vector demand is resolved into a heading angle, pitch attitude, and speed demand, which are passed to the control system. The superposition of these three rules results in all agents moving as a flock. A theoretical analysis of the emergent dynamics of flocking behavior given in [4] was performed.

A theoretical analysis of the emergent dynamics of flocking behavior given as follows. Each agent recognizes two physical values: 1) the distance to its nearest flockmates and 2) the relative velocity of its flockmates. Agent i sees a neighboring agent j in its visual sensor range. Agent i can recognize the vector $\boldsymbol{d}_{ij}$, that is, the position vector to the neighbor agent j and can calculate the relative center position vector $\boldsymbol{D}_i$ of the neighboring flockmates. Agent i also recognizes vector $\boldsymbol{v}_{ij} = d\boldsymbol{d}_{ij}/dt$, that is, the relative velocity vector and can calculate the average relative velocity vector $\boldsymbol{V}_i$ of the neighboring flockmates. The relative center position vector $\boldsymbol{D}_i$, its unit vector $\boldsymbol{e}_{\boldsymbol{D}_i}$, its size D_i, the average relative velocity vector $\boldsymbol{V}_i$, its unit vector $\boldsymbol{e}_{\boldsymbol{V}_i}$ and its size V_i,are:

$$\begin{cases} \boldsymbol{D}_i = \dfrac{1}{n_i}\displaystyle\sum_{j}^{n_j}\boldsymbol{d}_{ij} \ , D_i = |\boldsymbol{D_i}| \ , \boldsymbol{e}_{\boldsymbol{D_i}} = \dfrac{\boldsymbol{D_i}}{D_i} \\ \boldsymbol{V}_i = \dfrac{1}{n_i}\displaystyle\sum_{j}^{n_j}\boldsymbol{v}_{ij} \ , V_i = |\boldsymbol{V_i}| \ \ , \boldsymbol{e}_{\boldsymbol{V_i}} = \dfrac{\boldsymbol{V_i}}{V_i} \end{cases} \tag{1}$$

A linear combination of the cohesion force vector $\boldsymbol{F}_{ci}$, separation force vector $\boldsymbol{F}_{si}$, and alignment force vector $\boldsymbol{F}_{ai}$ are used to define the flocking force vector $\boldsymbol{F}_{fi}$.

$$\begin{cases} \boldsymbol{F}_{ci} = w_{ci}\boldsymbol{e}_{\boldsymbol{D}_i} \\ \boldsymbol{F}_{si} = -\dfrac{w_{si}}{D_i}\boldsymbol{e}_{\boldsymbol{D}_i} \\ \boldsymbol{F}_{ai} = w_{ai}\boldsymbol{V}_i \end{cases} \tag{2}$$

$$\boldsymbol{F}_{fi} = \boldsymbol{F}_{ci} + \boldsymbol{F}_{si} + \boldsymbol{F}_{ai} = \left(w_{ci} - \frac{w_{si}}{D_i}\right)\boldsymbol{e}_{\boldsymbol{D}_i} + w_{ai}\boldsymbol{V}_i \tag{3}$$

where coefficient w_{ci}, w_{si} and w_{ai} are positive. The first term of Eq.(3) is the resultant force of the cohesion force vector $\boldsymbol{F}_{ci}$ and the separation force vector $\boldsymbol{F}_{si}$. The resultant force vector $\boldsymbol{F}_{csi}$ relates position between agents.

$$\boldsymbol{F}_{csi} = \left(w_{ci} - \frac{w_{si}}{D_i}\right)\boldsymbol{e}_{D_i} \tag{4}$$

The potential ϕ_{csi} of $\boldsymbol{F}_{csi}$ is given by the following equation.

$$\phi_{csi} = \int |\boldsymbol{F}_{csi}| \, dD_i = w_{ci} D_i - w_{si} log(D_i) \tag{5}$$

The potential energy ϕ_{csi} has a local minimum at

$$D_i = \frac{w_{si}}{w_{ci}} \tag{6}$$

At this point, the force vector ϕ_{csi} equals the zero vector. When the distance D_i from the center of neighbors is less than the value of right side of Eq.(6), the force vector $\boldsymbol{F}_{csi}$ becomes repulsive. Otherwise, the force vector $\boldsymbol{F}_{csi}$ becomes attractive. If w_{si} is smaller or w_{ci} is larger, then the absolute value of the distance D_i from the center of neighbors becomes shorter. The second term in Eq.(3), the alignment term, aligns the velocity of each agents.

3 Complex Networks

Recently there has been a surge of interest in trying to understand the properties of realistic networks. The small-world model of Watts and Strogatz [5] caused a tremendous amount of interest among researchers working in multiple fields on the topological properties of complex networks. And another type of networks are particularly known as scale-free and different models for generating or growing them have been imagined. Still, one of the most important models for generating power-low networks is the preferential attachment growth, introduced by Barabási & Albert (BA)[6]. Another case of regular networks are lattices, which have evenly spaced nodes, each connected only to the nearest neighbors. An interesting property of lattices is that, although they do not display the small-world property, by taking a small number of randomly chosen links and rewiring them to another node, small-world properties emerge very quickly.

In most engineering and biological complex systems, the nodes are dynamic, that is, "real-life" engineering networks are interconnections of dynamic systems. Synchronization is the most prominent example of coherent behavior, and is a key phenomenon in systems of coupled oscillators as those characterizing most biological networks or physiological functions [7]. Synchronous behavior is also affected by the network structure. The range of stability of a synchronized state is a measure of the system ability to yield a coherent response and to distribute information efficiently among its elements.

4 Consensus, Synchronization and Graph Laplacian

In networks of dynamic agents, consensus means to reach an agreement regarding a certain quantity of interest that depends on the state of all agents. A consensus algorithm is an interaction rule that specifies the information exchange between an agent and all of its neighbors on the network. Olfati-Saber [8] demonstrates a phase transition phenomenon in algebraic connectivity of network and show good convergence speed for consensus problems on small-world networks. Each agent has cohesion, separation, and alignment (CSA) rules that were originally introduced by Reynolds [1]. Each agent inputs the relative velocity and position of neighboring agents in its visual range and computes its steering and driving acceleration at that time. In other words, each agent has a local-directed link to other agents and the emerging flocking behavior. However, if agents are far from each other, flocking behavior cannot emerge.

A theoretical framework for the design and analysis of flocking algorithms for mobile agents was developed by Olfati-Saber [9] as a consensus problem. They demonstrated that flocks are networks of dynamic systems with a dynamic topology. This topology is a proximity graph that depends on the states of all the agents and is determined locally for each agent.

In networks of agents, consensus means to reach an agreement regarding a certain quantity of interest that depends on the state of all agents. A consensus algorithm is an interaction rule that specifies the information exchange between an agent and all of its neighbors on the network.

The analysis of consensus problems relies heavily on matrix theory and spectral graph theory. The interaction topology of a network of agents is represented using a directed graph G with the set of nodes and edges. We denote neighbors of agent i with n_i.

Consider a network of agents with the following dynamics:

$$\dot{x}_i = \sum_{j \in N_i} \alpha_{ij} \left(x_j(t) - x_i(t) \right) \tag{7}$$

Here, reaching a consensus means asymptotically converging to the same internal state by way of an agreement characterized by the following equation:

$$x_1 = x_2 = \cdots = x_n = \alpha \tag{8}$$

The collective dynamics converge to the average of the initial states of all agents:

$$\alpha = \frac{1}{n} \sum_{i=1}^{n} x_i(0) \tag{9}$$

The dynamics of system in Eq. 7 can be expressed as

$$\dot{\mathbf{x}} = -\mathbf{L}\mathbf{x}(t) \tag{10}$$

$\mathbf{L}$ is the graph Laplacian of the network G; the graph Laplacian is defined as

$$\mathbf{L} = D - A \tag{11}$$

where $D = \mathrm{diag}(d_1, d_2, \cdots, d_n)$ is the diagonal matrix with elements $d_i = \sum_{j \neq i} a_{ij}$.

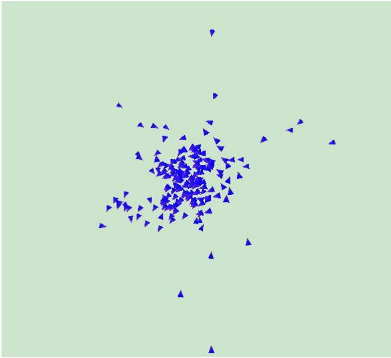

Fig. 1. Snapshot of simulation. $p = 10^{-3}$, time step $t = 100$.

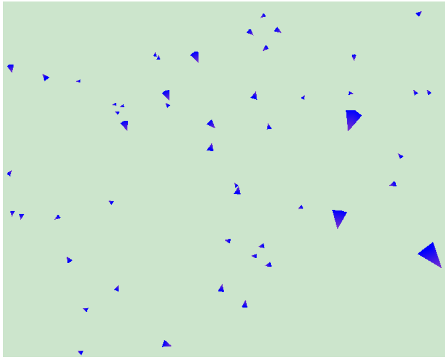

Fig. 2. Snapshot of simulation. $p = 10^{-4}$, time step $t = 100$.

The Laplacian matrix of a graph is defined as $L = D - A$, where D is the degree matrix, and A is the adjacency matrix [10].

Observation to make is that $\lambda_1 = 0$ for any Laplacian matrix, because $[1 \cdots 1]^T$ is an eigenvector. Since any given row's diagonal entry is k_i, and there is a -1 for each connection (a total of k_i of them), we have

$$\begin{bmatrix} k_1 & & \{0,-1\} \\ & k_2 & & \\ & & \ddots & \\ \{0,-1\} & & & k_n \end{bmatrix} \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} = 0 \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} \tag{12}$$

The eigenvalues λ_i satisfy

$$0 = \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n \leq 2k_n, \tag{13}$$

where k_n is the largest degree in the graph[11].

A graph with a small first Laplacian eigenvalue, λ_2, customarily called algebraic connectivity (or also spectral gap) has a relatively clean bisection. In other words, the smaller the algebraic connectivity the smaller the relative number of links required to be cut-away to generate a bipartition. Conversely a large algebraic connectivity characterizes non-structured networks, with poor modular structure, in which a clear-cut separation into subgraphs is not inherent, i.e. in general higher λ_2 indicates graphs with smaller diameter and higher connectivity.[12] This is a crucial topological implication of algebraic connectivity. The best synchronizability is obtained by maximizing the algebraic connectivity.

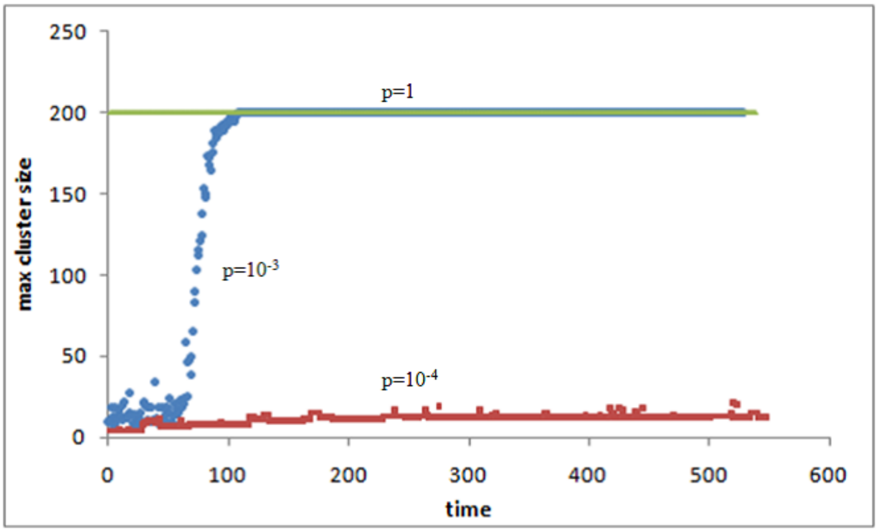

Fig. 3. The growth process of the largest agent cluster connected with local links. Vertical line is the number of agents that consist the largest cluster size, the total number of agents ($N = 200$).

5 Simulation Results

5.1 Simulation Condition

A simulation was performed where the state of the emerging flocking behavior depended on the probability of the external directed link p. If $0 < p < 1$, then the group of agents are in a small-world network. The probability for an external link was set to $p = 1$, 10^{-3} or 10^{-4} for the simulation.

At initial state (time step $t = 0$), 200 agents are randomly deployed in a sphere with radius 100. The visual range of each agent is a sphere with radius 3. It is the condition that most of the agents are too far apart to be locally linked except for nonsignificant a few agent pairs. The velocity vector of each agent has a random magnitude on closed interval $[0, 0.2]$ and a random direction.

Each agent accelerates 0.2 times more than current velocity by a simulation step. Therewith it generates the acceleration from the force of Cohesion, Separation and Alignment rules as shown Eq.(3). And the magnitude of velocity vector is limited a maximum of 0.2. If there are not at all agents within its visual range of an agent, then it goes straight at an accelerated rate to the direction of its initial velocity. When its speed reach to the maximum, its motion transits uniform motion.

5.2 Growth of the Agent Cluster Network

Figure.1 shows a snapshot of the simulation where each agent has a probability for external link $p = 10^{-3}$ at a time step $t = 100$. It shows emergent flocking

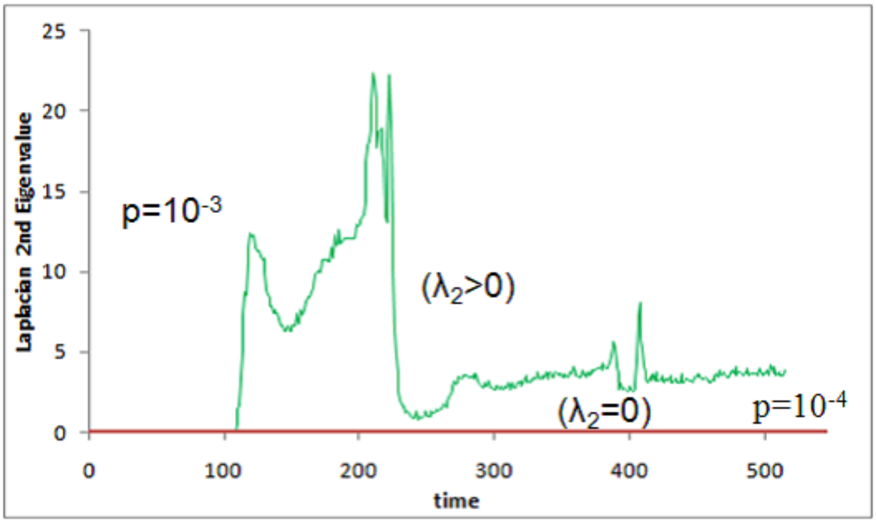

Fig. 4. The change of Laplacian 2nd smallest eigenvalue over time.

behavior. Figure.2 shows that the agents disengage from the flock and, thus, flocking behavior cannot occur. This implies that a group of agents cannot form a flock unless there are enough external links ($p > 10^{-3}$). Agents aggregate and form agent cluster with local links. Figure.3 shows the growth process of largest agent cluster connected with local links excluding external links. A few pairs consist as it happens at time $t = 0$.

In the case where $p = 10^{-4}$, some agents happen to close and to encounter. But they are disconnected by lapse of time. This means that stochastic external links contribute to break up rather than to aggregate.

In the case where $p = 1$, agents aggregate each other and agent cluster growth. All agent are connected and form an agent cluster. After that time, all agents continuity is connected. In other words, the connectivity is completely steady.

In the case where $p = 10^{-3}$, agents aggregate each other and agent cluster growth. It causes delay as compared with the case of $p = 1$. Almost all the agents are connected at time around $t = 100$, and the connectivity is not completely steady but slightly has a fluctuation. This means that stochastic external links sometimes contribute to break up rather than to aggregate.

5.3 Networked Consensus Agents Dynamics through Lapalcian Matrix Eigenvalue

Each agent aggregates together based on the cohesion and separation rules. Each agent adjusts its velocity using the alignment rule, and emergence of flocking behavior occurs. We approach the network dynamics by Lapalcian eigenvalue

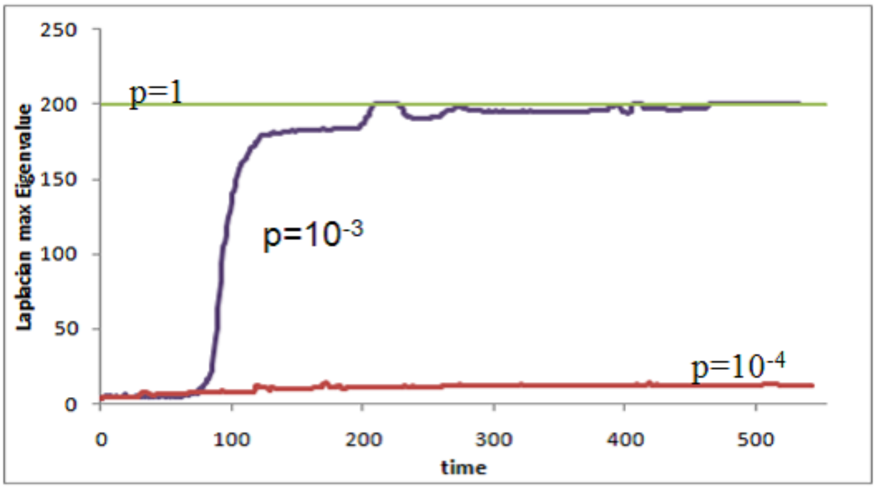

Fig. 5. The change of Laplacian max eigenvalue over time.

viewoint. In networks of agents, consensus means to reach an agreement regarding a certain quantity of interest that depends on the state of all agents. A consensus algorithm is an interaction rule that specifies the information exchange between an agent and all of its neighbors on the network. Flocking behavior is one of the consensus problem things. The emergence of Flocking behavior need to reach an agreement regarding a certain quantity such as velocity, direction and interaction rule that specifies the information exchange between an agent and all of its neighbors on the network.

The analysis of consensus problems relies heavily on matrix theory and spectral graph theory,and Laplacian matrix eigenvalues play an important role to think about the problems. Figure.4 shows the 2nd Laplacian eigenvalue transition of the flocking agents' network. Figure.5 shows the max Laplacian eigenvalue transition of the flocking agents' network. By these eigenvaules time transition dynamics, we can understand the network dynamics.

2nd Laplacian eigenvalue, λ_2, customarily called algebraic connectivity (or also spectral gap) has a relatively clean bisection. Crucial topological implication is a large algebraic connectivity characterizes non-structured networks, with poor modular structure, in which a clear-cut separation into subgraphs is not inherent, i.e. in general higher λ_2 indicates graphs with smaller diameter and higher connectivity. By 2nd eigenvalue transition of Figure.4, we can grasp the network dynamics. Max Laplacian eigenvalue,λ_n expresses robustness to information time delays, and λ_n is relational to the network max cluster size. If $\lambda_2 = 0$, the flocking agents' clusters are separated, and the network is disconnected. But, if $\lambda_2 > 0$ the flocking agents' clusters are not separated, the network is connected.

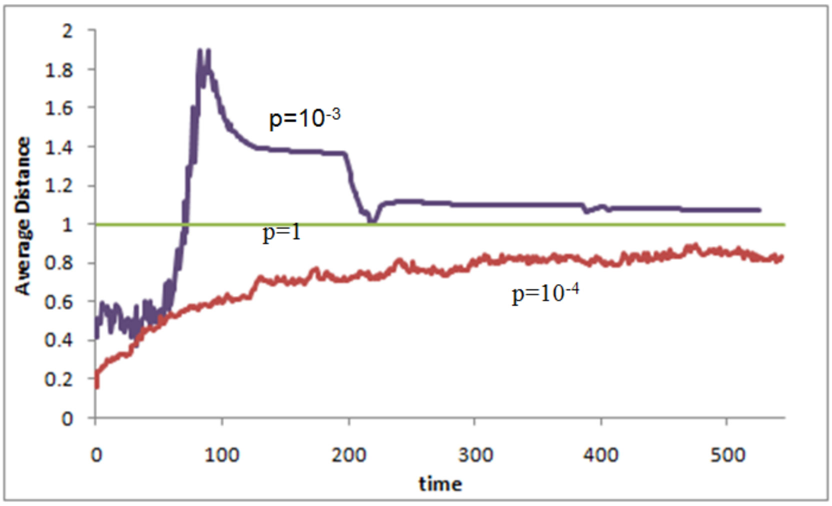

Fig. 6. The change of average distance over time.

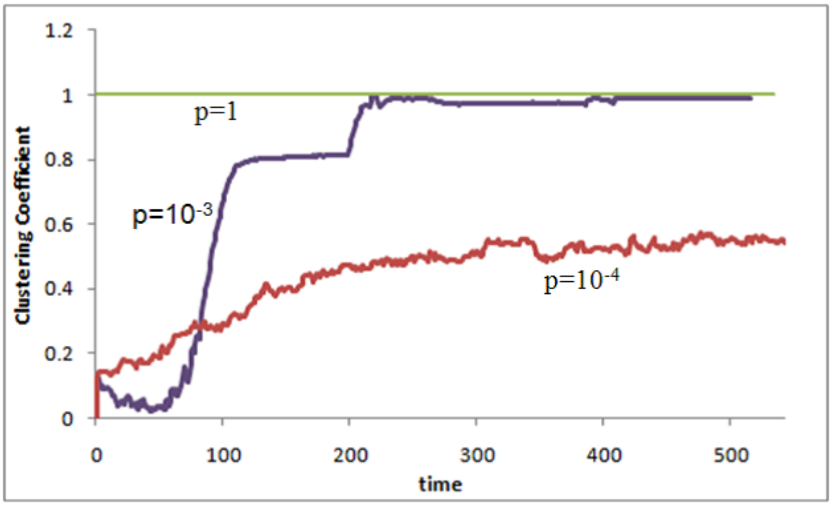

Fig. 7. The change of clustering coefficient over time.

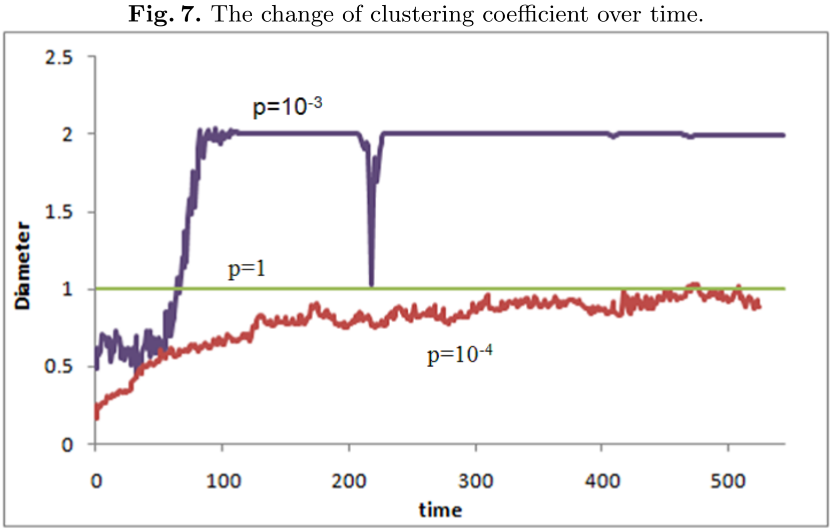

Fig. 8. The change of diameter over time.

In the case where $p = 1$, all agents always link directly each other. Thus, the agents can aggregate, so λ_2 and λ_n is always N(All number of nodes or agents) In the case where $p = 10^{-4}$, each agent never has enough links and, thus, the agents are dispersed. The flocking agents' clusters are separated, and the network is disconnected, so λ_2 is always 0. And flocking agents' clusters are small so that λ_n is kept small value all the time. In the case where $p = 10^{-3}$, all agents always have enough links to flock together. However, the agents as a group undergo a state transition from dispersing to aggregating. From the first point, $\lambda_2 > 0$ time, λ_2 increases gradually. This means that flocking network's connectivity is higher and diameter is smaller as λ_2 increases. And λ_2 reaches highest peak, flocking agents reach an agreement regarding a certain quantity such as velocity, direction. In other words, the network has achieved consensus. After that, λ_2 drastically drops and keeps low value. This means that flocking agents reach an agreement and consensus, and network do not need high connectivity and consensus state continue.

λ_2 and λ_n dynamics correspond to the diameter, average distance and clustering coefficient. The diameter D of a network is the longest are among all shortest paths between any pairs of nodes. It provides an upper bound on the average distance d_{ave}, which is the average length of all shortest paths between pairs of nodes. Given a node i, with k_i neighbours, E_i is defined to be number of links between the k_i neighbours. The clustering coefficient is the ratio between the number of links that exist between neighbours (E_i) and the potential number of links $k_i(k_i - 1)/2$ between the neighbours. The average clustering coefficient is defined as

$$CC = \frac{1}{N} \sum_{i=1}^{N} \frac{2E_i}{k_i(k_i - 1)} \tag{14}$$

Figure.6,7,8 shows average distance, clustering coefficient, diameter transition of the flocking agents' network. These network dynamics correspond to the Laplacian 2nd and max eigenvalue for the agent over time.

Stability is shown to rely on the connectivity properties of the graph that represents agent interconnections, in terms of asymptotic convergence with respect to arbitrary changes in the interconnection topology. We investigate the case where the topology of the control interactions between the agents is not fixed. The network topology property is dynamic and time varying. Exploiting results from algebraic graph theory, these properties are directly related to the topology of the network through the eigenvalues of the Laplacian of the graph. In this way, from the standpoint of the network eigenvalues viewpoint, we can understand the network dynamics structure.

6 Conclusion

The occurrence of flocking behavior in the networked agents is associated with the connectivity properties of the interconnection network. The network represents a dynamical systems which are coupled according to the network topology.

We consider the network by exploiting the graph theoretic properties of the agent networks. Our network showed how flocking behavior can emerge and converge to stability using a few external links, in addition to the local links among the autonomous agents. The topology of the network that allows observation of the shift in the network-wide patterns over time is important for robustness, since each agent needs to recognize not only its neighboring ones but also distant agents.

The information interaction for the agents is the behaviors of smooth and the stability analysis is based on the invariant principle, facilitated by the algebraic properties of the interconnection graph that allow the connectivity properties of the network to be reflected on the convergence estimate. Networked multi-agent systems are comprised of many autonomous interdependent agents which are decentralized group formation found in a complex network. The time varying nature of the interconnection topology introduces discontinuities in the control inputs, which in turns give rise to a set of discontinuous differential equations describing the system dynamics. As in the smooth case, the connectivity properties of the graph are instrumental in establishing global asymptotic stability. We can think about the dynamic networks and topology through algebraic graph theory, these properties are directly related to the eigenvalues of the Laplacian of the graph.

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