BOUNDEDLY RATIONAL EQUILIBRIUM AND RISK PREMIUM

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ABSTRACT. When people agree to disagree, how the disagreement among agents impact on the market is the main concern of this paper. With the standard mean variance framework, this paper consider a market of two risky assets and two agents who have different preference and heterogeneous beliefs in the mean and variance/covariance of the asset returns. By constructing a consensus belief, the paper develops an concept of boundedly rational equilibrium (BRE) to characterize the market equilibrium and examines explicitly the impact of heterogeneity on the market equilibrium and risk premium when the disagreements among the two agents are mean preserved spreads of a benchmark homogeneoue belief. It shows that the biased mean preserved spreads in beliefs among the two agents have significant impact on the risk premium of the risky assets and market portfolio in market equilibrium, and adding a riskless asset in the market magnifies the impact of the heterogeneity on the market. The results shed light on the risk premium and riskfree rate puzzles.

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1. INTRODUCTION

To better explain empirical anomalies, puzzles and various market phenomena, economics and finance are witnessing an important paradigm shift, from a representative, rational agent approach towards a behavioral, agent-based approach in which economy and markets are populated with bounded rational and heterogeneous agents. When the heterogeneity in beliefs is not due to asymmetric information but rather to intrinsic differences in how to view the world, people agree to disagree. The heterogeneity in beliefs among agents are very often characterized by notions of optimism and pessimism, overconfidence and doubt. Literatures have made a significant contribution to the understanding of the market aggregation when agents differ on their expectations and the impact of heterogeneous beliefs amongst investors on market equilibrium, see, for example, Lintner (1969), Rubinstein (1976), Williams (1977), Abel (1989, 2002), Detemple and Murthy (1994), Zapatero (1998)), and more recently Calvet, Grandmont and Lemaire (2004), Jouni and Napp (2006), Sharpe (2007)), Gollier (2007), and Chiarella, Deici and He (2006).

The notion of overconfidence has been explored in finance literature (see, e.g., DeLong, Shleifer, Summers and Waldmann (1990), Kyle and Wang (1997), and Daniel, Hirshleifer and Subrahmanyam (1998)). Several empirical studies of professionals' economic forecasts and psychological surveys indicate that agents have optimism and overconfidence about their own (relative) abilities (see, e.g., Rabin (1998), Hirshleifer (2001), and Giordani and Soderlind (2006)). As elaborated in Hvide (2002), experimental psychologic literature have applied, somewhat confusingly, two distinct meanings of the term 'overconfidence', overconfidence$_1$ and overconfidence$_2$ (called in Hvide (2002)). In the stock market, overconfidence$_1$ relates to a skewed first moment of a subjective probability distribution, while overconfidence$_2$ relates to a skewed second moment of a subjective probability distribution. The point is that there is no clear relation between overconfidence$_1$ and overconfidence$_2$ since they reflect different underlying phenomena. In Abel (2002), a uniform pessimism is defined as (the subjective distribution being) a leftward translation of the objective distribution, doubt as a mean-preserving spread of the objective distribution. To avoid confusion, in our discussion, we adopt the notions of Abel (2002) and refer overconfidence$_1$ and overconfidence$_2$ to optimism and overconfidence, respectively. The confidence in DeLong et al. (1990) and Kyle and Wang (1997) is actually referred to overconfidence$_2$.

One of the driving forces for the development of the literature in heterogeneous beliefs is to explain equity premium and risk-free rate puzzles, to which several theoretic explanation have been proposed recently. For example, Barberis, Huang and Santos (2001) adopt a non-standard utility function, motivated by prospect theory; Benartzi and Thaler (1995) consider myopic loss aversion. Deviating from rational expectation in the standard neoclassical paradigm, when beliefs are exogenously given, it has been found (see, e.g., Detemple and Murthy (1994), Abel (2002), Calvet et al. (2004), Jouini and Napp (2006), and Gollier (2007)) that a pessimistic bias and doubt in the subjective distribution of the growth rate of consumption and a positive correlation between risk tolerance and pessimism (doubt) leads to an increase of the market price of risk. To discipline the heterogeneity in beliefs and to understand how agents form their beliefs differently, in a static Nash equilibrium in demands (see Kyle (1989)) set up of two agents model when agents hold incorrect but strategic beliefs, Jouini and Napp (2009) provide a discipline for belief formation through a model of subjective beliefs in order to provide a rational for belief heterogeneity. They find that optimism (overconfidence) as well as pessemism (doubt) emerge as optimal beliefs of agents' strategic behaviour and there is a positive correlation between pessimism (doubt) and risk tolerance. This strategic explanation of heterogeneous beliefs is in contrast with rational approaches to beliefs where agents try to reflect the 'world as it is' in their beliefs, and with approaches in which forward-looking agents optimally distort beliefs and in which beliefs are of intrinsic value to agents, as with wishful thinking or fear of disappointment (see Brunnermeier and Parker (2005)).

However, by introducing a concept of pragmatic beliefs, Hvide (2002) use a simple game-theoretic example of a job market and show optimism can be the equilibrium outcome if agents form beliefs pragmatically. The main justification for programmatic beliefs is dynamics in the sense that, without awareness about their own optimism, agents are gradually learning that a certain way of forming beliefs is more rewarding than other ways. Also, by quantifying the amount of pessimism and doubt in survey data on US consumption and income, Giordani and Soderlind (2006) find some evidence of pessimism, but individual forecasters clearly exhibit overconfidence rather than doubt. By showing that the average distribution shows no statistically significant sign of either overconfidence or doubt, they conclude that doubt is not a promising explanation of the equity premium puzzle and the amount of pessimism provides only a rather small improvement in the empirical performance of the model.

The inconsistence between the theoretic and experimental or empirical results illustrates the complexity of the heterogeneity in beliefs and indicates a need to explicitly examine the complicated impact of the heterogeneity on the market aggregation. To this need, in this paper, we consider a simple financial market with two risky assets, one risk-free asset, and two agents who have different preferences and heterogeneous beliefs. Both agents have homogeneous beliefs in the return of the first asset which is well informed and understood, but they have heterogeneous beliefs in the return of the second asset. The heterogeneity of agents is characterized by their difference in different dimension, including risk tolerance, the expected returns, the standard deviations and the correlations of the two asset returns. The biased beliefs are assumed to be mean preserved spreads of some benchmark homogeneous belief. By assuming that agents maximize a primitive utility function (see Sharpe (1991) and Levy and Markowitz (1979)), agents choose their optimal portfolios based on their beliefs. By constructing a consensus belief, the market equilibrium is characterized by the consensus belief. Different from the standard rational expectation equilibrium, the market equilibrium under the consensus belief reflect the bounded rationality of the agents in the sense that the market equilibrium is achieved when agents make their optimal decision based on their subjective heterogeneous beliefs and therefore we call the equilibrium as a bounded rational equilibrium. We show that the different dimension in the biased beliefs can have different impact on the market equilibrium, in particular, the impact of heterogeneous beliefs in return correlation (which has not been examined in the literature since most of them consider the situation of only one risky asset).

The paper is structured as follows. In Section 2, we set up the economy and describe the aggregation problem when agents have heterogeneous preferences and beliefs. We show how to aggregate individual risk tolerance and individual beliefs through a consensus belief and obtain market equilibrium. In particular, we derive a CAPM under heterogeneous beliefs. In Section 3, as a benchmark of our analysis, we illustrate the traditional results under the homogeneous belief in terms of market portfolio, the equilibrium risk-free rate and market risk premium. In Section 4, we introduced biased risk preference and biased beliefs among two agents and examine explicitly the impact of the heterogeneity on the market equilibrium risk-free rate and market premium. As an application, we examine the conditions under which the risk-free rate and risk premium puzzles are reduced. Finally, some concluding remarks are presented in Section 5.

2. The Aggregation Problem, Consensus Belief, and Bounded Rational Equilibrium

2.1. **The Economy.** We consider a two-date economy in which there are two risky assets and two agents[1]. Let $\tilde{r}_j (j = 1, 2)$ be the return of the risky assets. Assume that there are two agents who have different preferences and heterogeneous beliefs in the expected return and variance-covariance of the asset return. For agent i $(i = 1, 2)$, let τ_i be his/her risk tolerance, and

$$\boldsymbol{\mu}_i = (\mu_{i,1}, \mu_{i,2})^T \qquad \text{and} \qquad V_i = \begin{pmatrix} \sigma_{i,1}^2 & \rho_i\sigma_{i,1}\sigma_{i,2} \\ \rho_i\sigma_{i,1}\sigma_{i,2} & \sigma_{i,2}^2 \end{pmatrix}$$

[1]The discussion and results of this section for the general economy of many risky assets and many heterogeneous beliefs can be found in He and Shi (2009)

be his/her beliefs in the expected returns and covariance matrix, respectively, where $\mu_{i,j} = \mathbb{E}_i(1+\tilde{r}_j), \sigma_{i,j}^2 = Var_i(\tilde{r}_j), \rho_i = Correl_i(\tilde{r}_1, \tilde{r}_2)$ for $i, j = 1, 2$. We use $\mathcal{B}_i := (\mu_{i,1}, \mu_{i,2}, \sigma_{i,1}, \sigma_{i,2}, \rho_i)$ to denote the beliefs of agent i.

2.2. **Portfolio Selection Problem.** We assume that investors maximize a primitive utility function $U_i(\boldsymbol{\pi}) = \boldsymbol{\pi}_i^T \boldsymbol{\mu}_i - \frac{1}{2\tau_i} \boldsymbol{\pi}_i^T V_i \boldsymbol{\pi}_i$ under the budget constraint is $\mathbf{1}^T \boldsymbol{\pi}_i = 1$, where $\boldsymbol{\pi}_i = (\pi_{i,1}, \pi_{i,2})^T$ is the vector of portfolio weights (proportion of wealth invested in each asset), τ_i is the risk-tolerance that measures the marginal rate of substitution of variance for expected return. This utility function has been used in Sharpe (1991), it is consistent with Markowitz portfolio selection criterion and also serves as a good approximation for other type of utility functions, see Levy and Markowitz (1979). There is the standard portfolio optimization problem and the optimal portfolio weights are given by

$$\boldsymbol{\pi}_i^* = \tau_i V_i^{-1}(\boldsymbol{\mu}_i - \lambda_i^* \mathbf{1}), \qquad \lambda_i^* = \frac{\mathbf{1}^T V_i^{-1} \boldsymbol{\mu}_i - 1/\tau_i}{\mathbf{1}^T V_i^{-1} \mathbf{1}} \tag{2.1}$$

where λ_i^* measures the marginal certainty equivalent rate of return (CER) per one percent investment in each asset.

In the case when there exists a risk-free security with a certain future payoff $R_f = 1 + r_f$, the CER of agents becomes R_f and the optimal portfolio can be simplified to

$$\boldsymbol{\pi}_i^* = \tau_i V_i^{-1}(\boldsymbol{\mu}_i - R_f \mathbf{1}). \tag{2.2}$$

2.3. **Consensus Belief and Bounded Rational Equilibrium.** We characterize market equilibrium by the concept of a consensus belief, which aggregates both investors' beliefs and determines the market equilibrium prices. We consider first the economy without a risk-free asset and then the economy with a risk-free asset.

Definition 2.1. *A belief $\mathcal{B}_a = (\mu_{a,1}, \mu_{a,2}, \sigma_{a,1}, \sigma_{a,2}, \rho_a)$, is called a market* **consensus belief** *if the equilibrium price vector of the risky assets (and the risk-free rate when there exists a risk-free asset) under the heterogeneous beliefs $\mathcal{B}_i := (\mu_{i,1}, \mu_{i,2}, \sigma_{i,1}, \sigma_{i,2}, \rho_i)$ $(i = 1, 2)$ is also the market equilibrium price vector of the risky assets (and the risk-free rate) under the homogeneous belief $\mathcal{B}_a$.*

Let W_{i0} be the initial wealth of agent $i\,(i = 1, 2)$. Then $W_0 = W_{10} + W_{20}$ corresponds to the total market wealth. Define the market wealth proportion $w_i = \frac{W_{i0}}{W_0}$ of agent $i\,(i = 1, 2)$, then the market clearing condition

$$W_o \boldsymbol{\pi}_m = W_{10} \boldsymbol{\pi}_1^* + W_{20} \boldsymbol{\pi}_2^*$$

becomes

$$\boldsymbol{\pi}_1^* \ w_1 + \boldsymbol{\pi}_2^* \ w_2 = \boldsymbol{\pi}_m,$$

where $\boldsymbol{\pi}_m$ denotes the market portfolio of risky assets. Since the market equilibrium is obtained based on the fact that both agents make their optimal portfolio decision under their subjective beliefs, rather than the objective belief, we call such equilibrium as *Boundedly Rational Equilibrium (BRE)*. The following result characterize such BRE.

Proposition 2.2. *Let*

$$\tau_a := w_1\ \tau_1 + w_2\ \tau_2, \qquad \lambda_a^* := w_1 \frac{\tau_1}{\tau_a}\ \lambda_1^* + w_2 \frac{\tau_2}{\tau_a}\ \lambda_2^*.$$

Then

(i) *the consensus belief $\mathcal{B}_a$ is given by*

$$V_a^{-1} = \frac{1}{\tau_a \lambda_a^*}\Big[w_1\tau_1\lambda_1^*\ V_1^{-1} + w_2\tau_2\lambda_2^*\ V_2^{-1}\Big], \tag{2.3}$$

$$\boldsymbol{\mu}_a = \mathbb{E}_a(\mathbf{1} + \tilde{\mathbf{r}}) = \frac{1}{\tau_a}\big[w_1\tau_1(V_1^{-1}V_a)\boldsymbol{\mu}_1 + w_2\tau_2(V_2^{-1}V_a)\boldsymbol{\mu}_2\big]; \tag{2.4}$$

(ii) *The equilibrium market portfolio is determined by*

$$\boldsymbol{\pi}_m = \tau_a V_a^{-1}(\boldsymbol{\mu}_a - \lambda_a^*\mathbf{1}); \tag{2.5}$$

(iii) *the Zero-beta CAPM relation*

$$\mathbb{E}_a[\tilde{\mathbf{r}}] - (\lambda_a^* - 1)\mathbf{1} = \boldsymbol{\beta}[\mathbb{E}_a(\tilde{r}_m) - (\lambda_a - 1)], \tag{2.6}$$

holds under the consensus belief $\mathcal{B}_a$, where

$$\boldsymbol{\beta} = (\beta_1, \beta_2)^T, \qquad \beta_j = \frac{Cov_a(\tilde{r}_m, \tilde{r}_j)}{\sigma_a^2(\tilde{r}_m)}, \qquad j = 1, 2.$$

Proposition 2.2 shows that the risk tolerance of the market is a weighted average of that of the two agents weighted by the corresponding wealth fraction of the agents, the consensus belief of the inverse variance/covariance matrix of asset return is an average of investors' subjective beliefs weighted by their wealth share, risk-tolerance and CER. Hence a wealthier and more risk-tolerant investor with a higher CER will dominate the consensus belief. The consensus belief of expected asset returns is also an average of investors' subjective beliefs weighted by their wealth share, risk-tolerance and also their confidence about future asset returns reflected by V_i^{-1}. Therefore, a wealthier, more risk-tolerant and confident investor will dominate the consensus belief of expected future asset returns.

If there exist a risk-free security, then $\lambda_i^* = R_f$ for all i in equations (2.3) and (2.4). In this case, the risk-free asset is in zero-net supply under market clearing conditions. Consequently, we have the following Corollary.

Corollary 2.3. *If there exist a risk-free security with return payoff of R_f, then the consensus belief $\mathcal{B}_a$ is given by*

$$V_a^{-1} = \frac{1}{\tau_a}\big[w_1\tau_1\, V_1^{-1} + w_2\tau_2\, V_2^{-1}\big], \tag{2.7}$$

$$\boldsymbol{\mu}_a = \mathbb{E}_a(\mathbf{1}+\tilde{\mathbf{r}}) = \frac{1}{\tau_a}\big[w_1\tau_1(V_1^{-1}V_a)\boldsymbol{\mu}_1 + w_2\tau_2(V_2^{-1}V_a)\boldsymbol{\mu}_2\big]; \tag{2.8}$$

the market portfolio becomes

$$\boldsymbol{\pi}_m = \tau_a V_a^{-1}(\mathbb{E}_a(\tilde{\mathbf{r}}) - r_f\mathbf{1}); \tag{2.9}$$

the CAPM relation becomes

$$\mathbb{E}_a[\tilde{\mathbf{r}}] - r_f\mathbf{1} = \boldsymbol{\beta}[\mathbb{E}_a(\tilde{r}_m) - r_f], \tag{2.10}$$

and the risk-free rate is given by

$$r_f = \frac{\mathbf{1}^T V_a^{-1}\mathbb{E}_a(\tilde{\mathbf{r}}) - \frac{1}{\tau_a}}{\mathbf{1}^T V_a^{-1}\mathbf{1}} \tag{2.11}$$

3. A Benchmark Case under Homogeneous Beliefs

To examine the impact of the heterogeneity on the market equilibrium and compare with the market equilibrium under a homogeneous belief, we consider in this section a benchmark case under the standard rational expectation in which both agents may have different risk tolerance, but have the same beliefs in return[2], denoted by $\mathcal{B}_o = (\mu_1, \mu_2, \sigma_1, \sigma_2, \rho)$, that is $\mathcal{B}_i = \mathcal{B}_o$ for $i = 1, 2$. In this benchmark case, we have from Proposition 2.2 that

$$V_a = \begin{pmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{pmatrix} := V_o, \qquad \boldsymbol{\mu}_a = (\mu_1, \mu_2)^T := \boldsymbol{\mu}_o.$$

Consequently, the market portfolio is simply given by

$$\hat{\boldsymbol{\pi}}_m = \frac{1}{\sigma_1^2+\sigma_2^2-2\rho\sigma_1\sigma_2}\big(\tau_a(\mu_1-\mu_2) + \sigma_2(\sigma_2-\rho\sigma_1), \quad \tau_a(\mu_2-\mu_1) + \sigma_1(\sigma_1-\rho\sigma_2)\big)^T. \tag{3.1}$$

When there exists a risk-free asset, the market risk-premium, risk-free return and market variance are respectively given by

$$\begin{aligned}
\hat{\mathbb{E}}(\tilde{r}_m - r_f) &= \frac{(\mu_1-\mu_2)^2\tau_a^2 + (1-\rho^2)\sigma_1^2\sigma_2^2}{\tau_a(\sigma_1^2+\sigma_2^2-2\rho\sigma_1\sigma_2)}, \\
\hat{R}_f &= \frac{\sigma_1^2\sigma_2^2}{\sigma_1^2+\sigma_2^2-2\rho\sigma_1\sigma_2}\left(\frac{\mu_1}{\sigma_1^2} + \frac{\mu_2}{\sigma_2^2} - \frac{\rho(\mu_1+\mu_2)}{\sigma_1\sigma_2} - \frac{1}{\tau_a}(1-\rho^2)\right), \\
\hat{\sigma}^2(\tilde{r}_m) &= \frac{(\mu_1-\mu_2)^2\tau_a^2 + (1-\rho^2)\sigma_1^2\sigma_2^2}{(\sigma_1^2+\sigma_2^2-2\rho\sigma_1\sigma_2)}
\end{aligned} \tag{3.2}$$

[2]The benchmark beliefs $\mathcal{B}_o$ can be treated as the objective beliefs. However, given that the consensus belief characterizes the market equilibrium, we would rather treat $\mathcal{B}_o$ as a benchmark belief.

And it is easy to see from above that $1/\tau_a = \hat{\mathbb{E}}(\tilde{r}_m - r_f)/\hat{\sigma}^2(\tilde{r}_m)$, so the market risk-tolerance is also the marginal rate of substitution between market risk premium and market variance.

Weil (1989) attributed the *risk premium puzzle* and the *risk-free rate puzzle* to two distinction basic empirical facts about the aggregate consumption process. (i) There is not enough individual consumption risk, so if agents are only moderately risk-averse, the observed risk premium is too high. The average rate of growth of individual consumption is too high, to explain the observed risk-free rate, agents need to be extremely averse to intertemporal substitution. Although in our model, investors consume everything on the liquidation day, we show that similar puzzles can still arise when asset returns are correlated and the asset with a higher expected future return is also much more risky. To illustrate this, we consider the following numerical example.

Example 3.1. *Let the two risky assets in the economy have expected returns* $(\mu_1, \mu_2) = (1.06, 1.09)$ *and standard deviations* $(\sigma_1, \sigma_2) = (0.08, 0.3)$ *and correlation coefficient* $\rho = 0.8$*. Both investors hold the benchmark belief, i.e* $\mathcal{B}_i = \mathcal{B}_o = (\mu_1, \mu_2, \sigma_1, \sigma_2, \rho)$*. For simplicity, we also assume that each investor has half of the aggregate market initial wealth (*$w_1 = w_2 = 1/2$*).*

We choose $\tau_i = 0.5\,(i = 1, 2)$, which is a reasonable level of risk-tolerance. This implies that the market in equilibrium requires 2% expected return for 10% standard deviation. Consequently, we have from equations (3.1) and (3.2) that the market portfolio is given by $\boldsymbol{\pi}_m = (0.962,\ 0.038)^T$ and

$$\tau_a = 0.5, \qquad \hat{r}_f = 4.62\%, \qquad \hat{\mathbb{E}}(\tilde{r}_m - r_f) = 1.49\%, \qquad \text{and} \qquad \hat{\sigma}_m = 8.63.$$

Clearly with $\tau_i = \tau_a = 0.5$, the risk-free rate is too high, also the risk-premium and market volatility is too low. We can certainly increase the risk-premium and reduce the risk-free rate by decreasing τ_a, however this leads to absurd implications. For example, if we choose $\tau_a = 0.1$, then

$$\hat{r}_f = 0\%, \qquad \hat{\mathbb{E}}(\tilde{r}_m - r_f) = 6\%, \qquad \text{and} \qquad \hat{\sigma}_m = 8\%$$

However, this means the market portfolio becomes $\boldsymbol{\pi}_m = (1,\ 0)^T$ (asset 2 is redundant), also the market requires 10% expected return for 10% standard deviation which doesn't seem reasonable and the market volatility is still too low. Therefore, it appears that merely assuming a lower risk-tolerance does not really solve the problem.

4. THE IMPACT OF HETEROGENEITY

In this section, we first assume that there is a risk-free asset and explore the impact of heterogeneity in investors' belief on the market equilibrium, in particular, the risk-premium and the risk-free rate. To examine the impact explicitly, we assume that agents agree about the first asset but disagree about the expected return, standard deviation of asset 2 and the correlation coefficient. That is, the beliefs in the expected return and the standard deviation of the first asset for both agents are given by the benchmark beliefs: $(\sigma_{i,1}, \mu_{i,1}) = (\sigma_1, \mu_1)$ for $i = 1, 2$, while the risk tolerance and beliefs in the return of the second asset of the two agent are mean-preserved spread of the benchmark belief. More precisely, we assume that the risk-tolerances of the two agents are given, respectively, by

$$\tau_1 = \tau_o(1 - \Delta), \qquad \tau_2 = \tau_o(1 + \Delta), \qquad \Delta \in (-1, 1);$$

the beliefs about the standard deviation of asset 2 are given by

$$\sigma_{1,2} = \sigma_2(1 - \delta), \qquad \sigma_{2,2} = \sigma_2(1 + \delta), \qquad \delta \in (-1, 1);$$

the beliefs about the correlation between asset returns are given by

$$\rho_1 = \rho(1 - \varepsilon), \qquad \rho_2 = \rho(1 + \varepsilon), \qquad \varepsilon \in (-1, 1);$$

and the beliefs of expected returns of asset 2 are given by

$$\mu_{1,2} = \mu_2(1 - \alpha), \qquad \mu_{2,2} = \mu_2(1 + \alpha), \qquad \alpha \in (-1, 1).$$

Hence, the "average" risk-tolerance and belief in this heterogeneous economy is exactly the same as the benchmark homogeneous economy. However, the consensus belief may not be same as the benchmark belief, as a result, the market portfolio, market risk-premium, risk-free rate and the market volatility may also differ from the homogeneous benchmark economy. For this setup, the heterogeneity is characterized by Δ, δ, ε and α. To examine the joint impact of different dimension of heterogeneity on the market, we consider three combinations of these parameters in the following.

4.1. **Case 1: Impact of Risk Tolerance and Optimism/Pessimism.** We first consider the case where the two agents have different risk-tolerance and also heterogeneous belief regarding the expected future return of asset 2, that is

$$\delta = 0, \qquad \varepsilon = 0, \qquad \Delta \in (-1, 1), \qquad \alpha \in (-1, 1). \tag{4.1}$$

Applying Corollary 2.3 to this case, we obtain the following result.

Corollary 4.1. *For the case* (4.1)*, the consensus belief is given by*

$$V_a = V_o \qquad \boldsymbol{\mu}_a = (\mu_1,\ \mu_2(1 + \alpha\Delta))^T \tag{4.2}$$

Consequently,

(i) *The change in market portfolio is given by*

$$\boldsymbol{\pi}_m - \hat{\boldsymbol{\pi}}_m = \frac{\alpha\Delta\ \tau_o\mu_2}{\sigma_1^2 - 2\rho\sigma_1\sigma_2 + \sigma_2^2}\big(- 1, 1\big); \tag{4.3}$$

(ii) *The change in risk-premium is given by*

$$(\mathbb{E}(\tilde{r}_m) - r_f) - (\hat{\mathbb{E}}(\tilde{r}_m) - \hat{r}_f) = \alpha\Delta\mu_2 \frac{\sigma_1(\rho\sigma_2 - \sigma_1) + \tau_o(\mu_2 - \mu_1)}{\sigma_1^2 - 2\rho\sigma_1\sigma_2 + \sigma_2^2}; \tag{4.4}$$

(iii) *The change in risk-free rate is given by*

$$\hat{r}_f - r_f = \alpha\Delta\sigma_1\mu_2 \frac{\rho\sigma_2 - \sigma_1}{\sigma_1^2 - 2\rho\sigma_1\sigma_2 + \sigma_2^2}; \tag{4.5}$$

(iv) *The change in market volatility is given by*

$$\sigma_m^2 - \hat{\sigma}_m^2 = \alpha\Delta\tau_o^2\mu_2 \frac{(\mu_2 - \mu_1) + (\mu_2(1 + \alpha\Delta) - \mu_1)}{\sigma_1^2 - 2\rho\sigma_1\sigma_2 + \sigma_2^2}; \tag{4.6}$$

(v) *The changes in the beta coefficients are given by*

$$\begin{aligned} \beta_1 &= \hat{\beta}_1 + \alpha\Delta\mu_2\tau_o \frac{\rho\sigma_2 - \sigma_1}{\sigma_1^2 - 2\rho\sigma_1\sigma_2 + \sigma_2^2}, \\ \beta_2 &= \hat{\beta}_2 + \alpha\Delta\mu_2\tau_o \frac{\sigma_2 - \rho\sigma_1}{\sigma_1^2 - 2\rho\sigma_1\sigma_2 + \sigma_2^2}. \end{aligned} \tag{4.7}$$

where $(\hat{\beta}_1,\ \hat{\beta}_2)^T = V_o\hat{\boldsymbol{\pi}}_m/\hat{\sigma}_m^2$ *are the asset betas under the homogeneous benchmark case.*

Corollary 4.1 characterize explicitly the impact of the heterogeneity on the market. It is easy to see that if either both the agents have the same risk preference (so that $\Delta = 0$) or they have the same benchmark beliefs in the expected return of both assets (so that $\alpha = 0$), then the results for the heterogeneous beliefs are reduced to that for the benchmark homogeneous case. The impact of the heterogeneity in this case (4.1) depends on the sign of $\alpha\Delta$ and the return correlation ρ in the benchmark belief. The condition $\alpha\Delta > 0$ implies that the more risk-tolerant investor is also optimistic about future asset return; that is the risk tolerance and optimism of agent are positively correlated. Similarly, $\alpha\Delta < 0$ implies that the less risk-tolerant investor is also optimistic about future asset return, so the risk tolerance and pessimism of agent are positively correlated. To simplify our discussion, we assume that $\mu_1 \le \mu_2$ and $\sigma_1 \le \sigma_2$. Based on the market consensus belief, Corollary 4.1 leads to following implications.

(i) When risk-tolerance and optimism about future returns are positively (negatively) correlated, that is $\alpha\Delta > (<)0$, it follows from (4.2) that the aggregate market is optimistic (pessimistic) about the expected return of the second asset. Consequently, the aggregate market, indicated by the market portfolio in (4.3), invests more (less) into asset 2 and less (more) into asset 1.

(ii) Comparing with the benchmark belief case, we have from (4.4) that the market with biased beliefs among the two agents increase the market risk-premium when either

$$\alpha\Delta > 0 \qquad \text{and} \qquad \mu_2 - \mu_1 > \sigma_1(\sigma_1 - \rho\sigma_2)/\tau_o \tag{4.8}$$

or

$$\alpha\Delta < 0 \qquad \text{and} \qquad \mu_2 - \mu_1 < \sigma_1(\sigma_1 - \rho\sigma_2)/\tau_o \tag{4.9}$$

.

From (4.5), the risk-free rate under the biased belief is reduced when either,

$$\alpha\Delta > 0 \qquad \text{and} \qquad \rho > \sigma_1/\sigma_2, \tag{4.10}$$

or

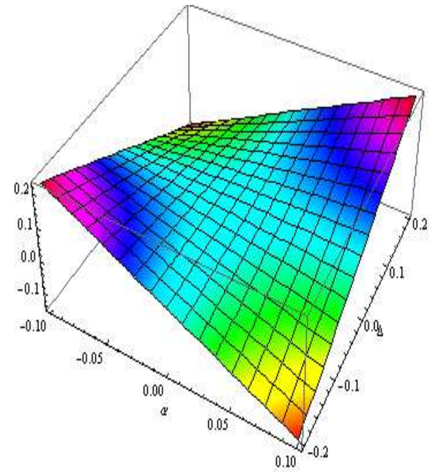
$$\alpha\Delta < 0 \qquad \text{and} \qquad \rho < \sigma_1/\sigma_2. \tag{4.11}$$

In other words, when either (i) the risk tolerance and optimism of agent are positively correlated, returns of the two assets are highly positively correlated (so that $\rho > \sigma_1/\sigma_2$), and also the difference in asset expected returns are large enough ($\mu_2 - \mu_1 > \sigma_1(\sigma_1 - \rho\sigma_2)/\tau_o$) or (ii) the risk tolerance and optimism of agent are negatively correlated, returns of the two assets are less (even negatively) correlated (so that $\rho < \sigma_1/\sigma_2$), and difference in asset expected returns are small enough ($\mu_2 - \mu_1 < \sigma_1(\sigma_1 - \rho\sigma_2)/\tau_o$), then the biased beliefs increase the market premium and reduce the risk-free rate. This analysis helps us to resolve the equity premium and risk-free rate puzzles under the standard benchmark homogeneous case.

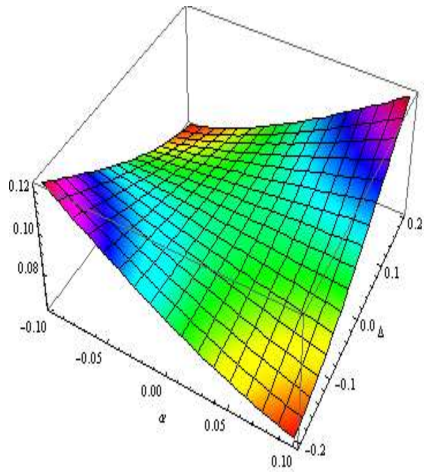
(iii) We have from (4.5) that the market volatility measured by σ_m is high (low) if $\alpha\Delta > (<)0$.

(iv) The standard CAPM relation under the benchmark belief no longer holds, though the CAPM still holds under the consensus belief. The betas under the biased beliefs can be decomposed into the betas under the benchmark belief and a term related to the biases in the beliefs. Comparing with the benchmark case, $\beta_1 > \hat{\beta}_1$ under either condition (4.8) or condition (4.9). However, $\beta_2 > \hat{\beta}_2$ whenever $\alpha\Delta > 0$.

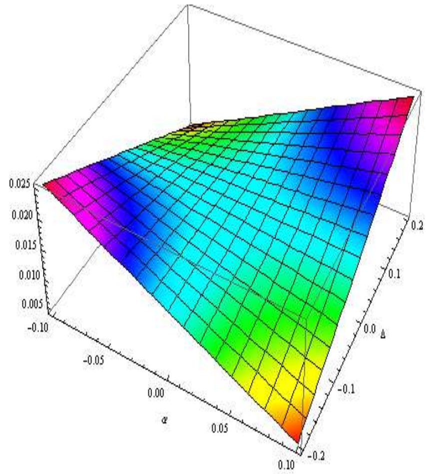
Based on the numerical values provided in Example 3.1, we are able to show graphically the impact of heterogeneity in terms of α and Δ on the change of market portfolio (in terms of the second risky asset in the market portfolio), market volatility, the expected market return and the equilibrium risk-free rate in market equilibrium when both α and Δ change in Figure 4.1. Note that $\rho > \sigma_1/\sigma_2$ for the numerical values. It is clear that the 3D plots are symmetric reflecting the fact the effect of heterogeneity depends on the product $\alpha\Delta$ rather than individually. We see in this case that as the product $\alpha\Delta$ gets larger, the market portfolio consists more of asset 2, which leads to higher market return and volatility, at the same time risk-free rate is also reduced which then increases the risk-premium. The Sharpe ratio of the market portfolio also increases, which suggests that heterogeneity of $\alpha\Delta$ improves the mean-variance efficiency of the aggregate market.



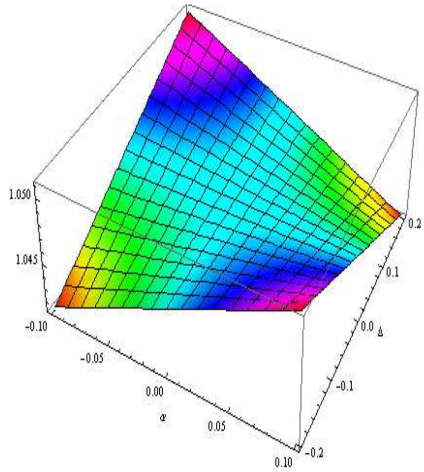
(a1) Market Proportion of asset 2



(a2) Market volatility



(a3) Risk premium



(a4) Risk-free rate

FIGURE 4.1. Effect of heterogeneity in risk-tolerance Δ and beliefs of expected return α on the market proportion of asset 2, market volatility, market risk-premium and the risk-free rate

Cases	$(\Delta, \delta, \varepsilon, \alpha)$	$\pi_{m,2}$	$\sigma(\tilde{r}_m)$	$\mathbb{E}(\tilde{r}_m - r_f)$	r_f	$\frac{\mathbb{E}(\tilde{r}_m)-r_f}{\sigma_m}$
Benchmark	$(0, 0, 0, 0)$	0.038	8.63%	1.49%	4.62%	0.1727
Case 1	$(0.2, 0, 0, 0.1)$	0.2258	12.3%	2.54%	4.14%	0.2061
Case 2	$(0, -0.2, 0, 0.1)$	0.7511	24.15%	3.88%	4.37%	0.1606
Case 3	$(0, 0, 0.2, 0.1)$	0.5415	19.31%	5.69%	1.94%	0.2947

TABLE 4.1. Effects of heterogeneity on the market proportion of asset 2 ($\pi_{m,2}$), market volatility ($\sigma(\tilde{r}_m)$), market risk-premium ($\mathbb{E}(\tilde{r}_m - r_f)$), the risk-free rate ($r_f$) and the Sharpe ratio $\frac{\mathbb{E}(\tilde{r}_m)-r_f}{\sigma_m}$ for the three cases, compared with the benchmark homogeneous case.

Based on the numerical values provided in Example 3.1, we choose $(\Delta, \delta, \varepsilon, \alpha) = (0.2, 0, 0, 0.1)$ in the first case and report the outcomes in Table 4.1. Comparing with the benchmark homogeneous belief, the results in Table 4.1 shows that heterogeneity in risk tolerance and expected return helps to resolve the puzzles when $\alpha\Delta > 0$, however, the overall effect is not too great for the chosen parameters. Risk premium increases moderately by 1% and the risk-free rate is merely reduced by less than half of a percent.

4.2. **Case 2: The Impact of Optimism/Pessimism and Confidence/Doubt.** In the second case, we focus on the impact of the optimism/pessimism (measured by α) and confidence/dobut (measured by δ) for asset 2 on the market in equilibrium and let $\Delta = 0,\ \varepsilon = 0$.

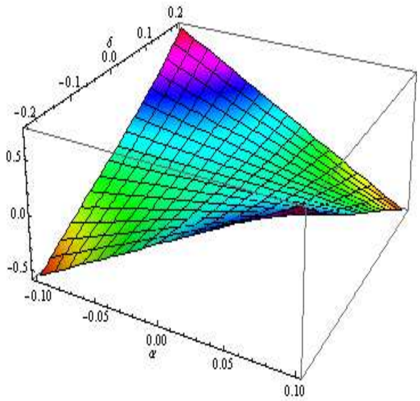
Corollary 4.2. *For the second case when $\Delta = 0,\ \varepsilon = 0$ and $\delta,\ \alpha \in (-1, 1)$, the consensus belief $\mathcal{B}_a = (\mu_{a,1}, \mu_{a,2}, \sigma_{a,1}, \sigma_{a,2}, \rho_a)$ is given by*

$$\mu_{a,1} = \mu_1 - \alpha\delta\mu_2 \frac{\rho\sigma_1}{\sigma_2(1+\delta^2-\rho^2)}, \qquad \mu_{a,2} = \mu_2\big(1 - \frac{\alpha\delta(2-\rho^2)}{1-\rho^2+\delta^2}\big). \tag{4.12}$$

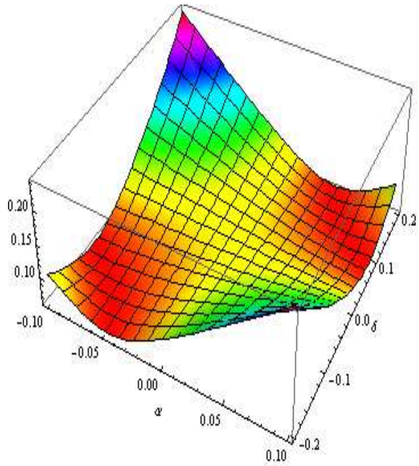
and

$$\sigma_{a,1}^2 = \sigma_1^2\big[1 - \frac{\delta^2\rho^2}{1+\delta^2-\rho^2}\big], \quad \sigma_{a,2}^2 = \sigma_2^2\frac{(1-\delta^2)^2(1-\rho^2)}{1+\delta^2-\rho^2}, \quad \rho_a = \rho\big[1 - \frac{\rho^2\delta^2}{1+\delta^2-\rho^2}\big]. \tag{4.13}$$

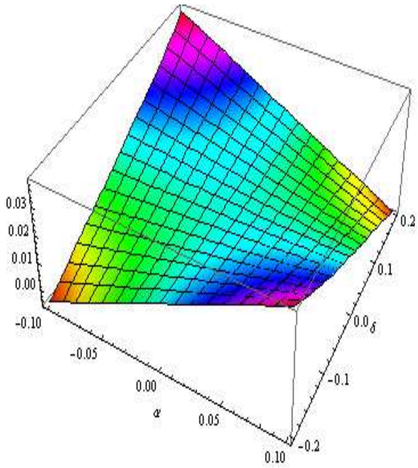
Corollary 4.2 gives the explicit impact of the biased beliefs in the expected return and the standard deviation for the second asset among the two agents. Two special cases are very interesting. The first one is that when there is unbiased beliefs in the standard deviation of the second asset (so that $\delta = 0$), there is no difference between the heterogeneous case with biased expected return on the second asset and the benchmark unbiased case, so a biased beliefs in the expected return of the asset 2 alone has no impact on the market. The second case is that the biased beliefs in the expected returns of the asset 2 has impact on the market expected return only when there is also a biased beliefs on the standard deviation of the return of the asset 2. In fact, from (4.12), the biased beliefs on the first and second moments of asset 2 can affect the market expected returns of both assets. It is not surprising to see that the biased beliefs in the standard deviation of the asset 2 affect the variances and covariance of both assets in aggregate. However, to our knowledge, this joint impact of the optimism/pessimism and confidence/dobut about one asset on the market expected returns of both assets has not been yet explored. From equations (4.13), one can see that the aggregate market becomes over-confident when investors have biased beliefs regarding the variance of asset 2's return so that, for $0 < \delta < 1$, $\sigma_{a,1} < \sigma_1,\ \sigma_{a,2} < \sigma_2$ and $\rho_a\sigma_{a,1}\sigma_{a,2} < \rho\sigma_1\sigma_2$. From (4.12), when $\alpha\delta < 0$, that is when the optimistic (pessimistic) investor is confident (doubtful) about his/her belief of the expected return of asset 2, the market expected return is higher for asset 2 than the benchmark expected return and is also higher for asset 1 when $\rho > 0$, but lower when $\rho < 0$. Intuitively, when $\rho > 0$, this would lead the aggregate market to invest more into the riskier asset, thus increasing the market return and volatility. This intuition is confirmed by Figure 4.2 where $\rho = 0.8$.



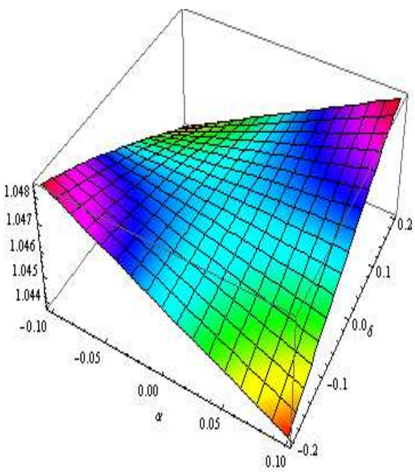
(b1) Market Proportion of asset 2



(b2) Market volatility



(b3) Risk premium



(b4) Risk-free rate

FIGURE 4.2. Effect of heterogeneity in beliefs of expected return α and variance of return δ on the market proportion of asset 2, market volatility, market risk-premium and the risk-free rate

Let $\delta = -0.2$ and $\alpha = 0.1$, that is the first (second) investor is optimistic (pessimistic) and confident (doubt) on the expected return of the second asset, so that $\alpha\delta < 0$. In this case, the results in Table 4.1 show a dramatic increase in the market's holding of asset 2, since it is the riskier asset with a higher expected return, therefore the market gains in risk-premium, however, also becomes much more volatile. The fact that the risk-free rate reduces only slightly means that the increase in risk-premium is mainly due to the increase in market expected return. Sharpe ratio drops comparing the homogeneous benchmark case, suggesting that the gain in risk premium cannot compensate for the higher volatility. Figure 4.2 shows 3D plots for different combinations of δ and α. The plots are symmetric, suggesting that the effect on equilibrium quantities depend on the product $\alpha\delta$ rather than individually.

4.3. **Case 3: The Impact of Optimism/Pessimism and Biased Belief in the Return Correlation.** In the third case, we examine the impact of heterogeneity in the expected return of asset 2 and the correlation coefficient. That is we let $\Delta = 0, \delta = 0$ and consider the effect of (ε, α) only.

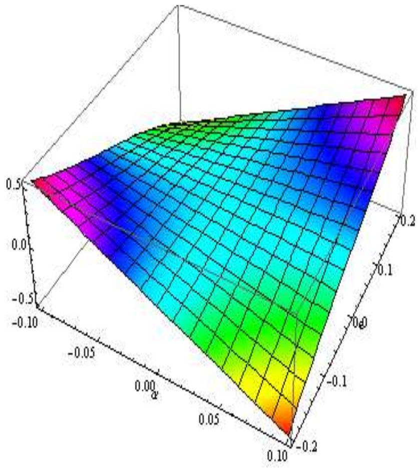
Corollary 4.3. *For the case that* $\Delta = 0, \delta = 0$ *and* $\varepsilon, \alpha \in (-1, 1)$*, the consensus belief* $\mathcal{B}_a = (\mu_{a,1}, \mu_{a,2}, \sigma_{a,1}, \sigma_{a,2}, \rho_a)$ *is given by*

$$\mu_{a,1} = \mu_1 - \alpha\varepsilon \frac{\rho\sigma_1}{(1-\rho^2)\sigma^2}\mu_2, \qquad \mu_{a,2} = \mu_2\big[1 + \alpha\varepsilon\frac{\rho^2}{1-\rho^2}\big], \tag{4.14}$$

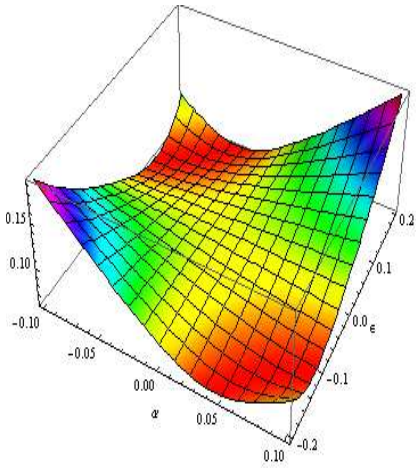
$$\sigma_{a,1}^2 = \sigma_1^2\big[1 - \frac{\varepsilon^2\rho^2}{1-\rho^2}\big], \quad \sigma_{a,2}^2 = \sigma_2^2\big[1 - \frac{\varepsilon^2\rho^2}{1-\rho^2}\big], \quad \rho_a = \rho\big[1 + \frac{\varepsilon^2\rho^2}{1-\rho^2}\big]. \tag{4.15}$$

Corollary 4.3 shows the impact of the optimism/pessimism and the biased beliefs in the correlation on the market. Note that the biased beliefs in the expected return of the asset 2 affect the market expected returns of both assets, not the variances and covariance. However, the biased beliefs in the return correlation affect both the first and second moments of the market returns of both assets. It is easy to see that, for $0 < \varepsilon < 1$, we have $\sigma_{a,1} < \sigma_1, \sigma_{a,2} < \sigma_2$ and $\rho_a > \rho$. This indicates that in aggregate the market becomes more confident about the future returns of the both assets and the returns of the both assets are highly correlated, comparing the benchmark case. For $\alpha\varepsilon > 0$, that is when the optimistic investor also believes in higher correlation between asset returns, we see from equation (4.14) that the market expected return for asset 2 is always higher, but lower (higher) when $\rho > 0 (< 0)$. Intuitively, this should lead the market to invest more into asset 2. For $\rho = 0.8$, Figure 4.3 illustrates the impact of the changes in δ and ϵ on the market, indicating higher equity premium and lower risk-free rate when the optimistic (pessimistic) investor believes higher (lower) correlation. The plots illustrate that the effect of heterogeneity is symmetric, the product $\alpha\varepsilon$ plays the crucial role in affecting the equilibrium quantities rather than individual parameters.

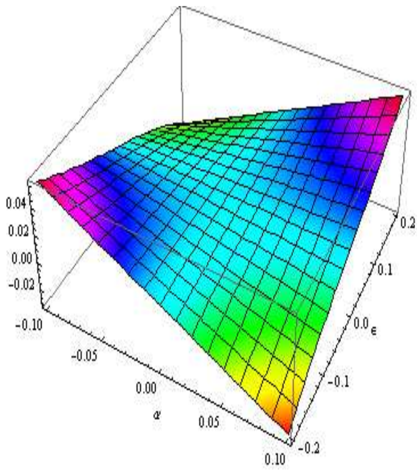
For given $\epsilon = 0.2$ and $\alpha = 0.1$, we have $\alpha\varepsilon > 0$. Table 4.1 gives the equilibrium outcomes, leading to the most desirable result. The risk-free rate in this case is reduced significantly by nearly 3% while the risk premium increased significantly by more than 4%. The market is more volatile, but less so than in the second case. Most noticeably, the Sharpe ratio in this example is 0.2497, highest amongst all cases including the homogeneous benchmark by far. This means the aggregate market is most mean-variance efficient when $\alpha\varepsilon > 0$.



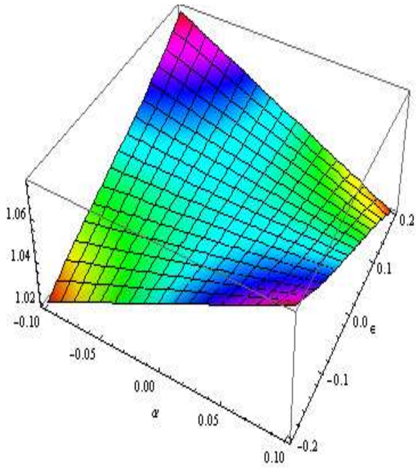
(c1) Market Proportion of asset 2



(c2) Market volatility



(c3) Risk premium



(c4) Risk-free rate

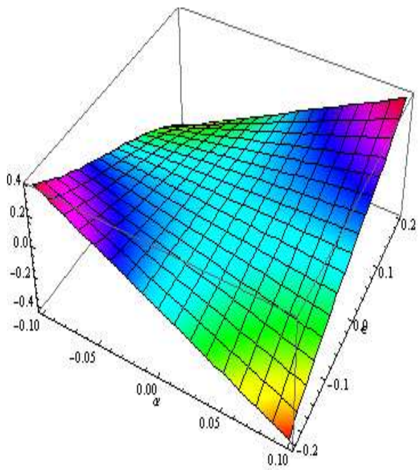
FIGURE 4.3. Effect of heterogeneity in beliefs of expected return α and the correlation coefficient ε on the market proportion of asset 2, market volatility, market risk-premium and the risk-free rate

4.4. **Impact of the existence of a risk-free asset.** In all of the above cases, we have assumed that there exists a risk-free security in the market, which is in net-zero supply to allow investors borrow and lend at the risk-free rate. According to equation (4.11) and (2.2), this implies that both investors have the same CER, that is $\lambda_i^* = R_f$ for $i = 1, 2$. If we there is no risk-free security in the economy, then obviously investors would have different CER and the aggregate market's CER is risk-tolerance weighted average of investors' CER. Note that when investors have common variance/covaraince matrices, then adding a risk-free security to the market would not change the consensus belief or the market equilibrium. This is, when $V_i = V_o$, the consensus belief of expected asset return are identical with or without a risk-free asset and therefore we have

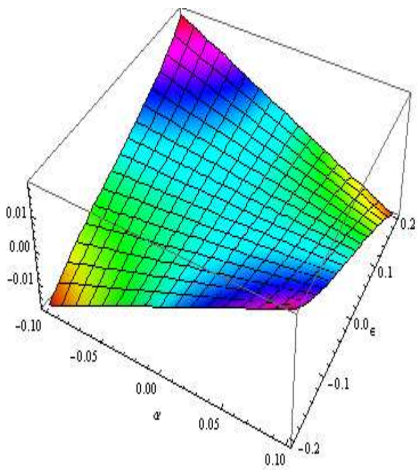
$$\lambda_a^* = R_f = \frac{\mathbf{1}^T V_a^{-1}\mu_a - \frac{1}{\tau_a}}{\mathbf{1}^T V_a^{-1}\mathbf{1}}$$

Hence, the aggregate market's CER would not be affected by the existence of a risk-free security in net-zero supply. However, this is no longer the case when investor have different beliefs about the varaiance/covariances of asset return. Market could arrive at a different equilibrium with or without a risk-free security.

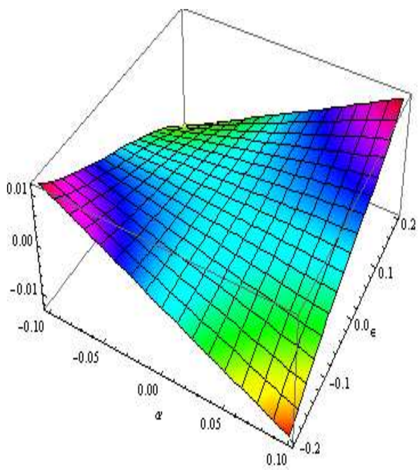
To illustrate the impact of the existence of the risk-less asset, we consider the Case 3, however, now assume that the market does not have a risk-free security available for borrowing or lending.



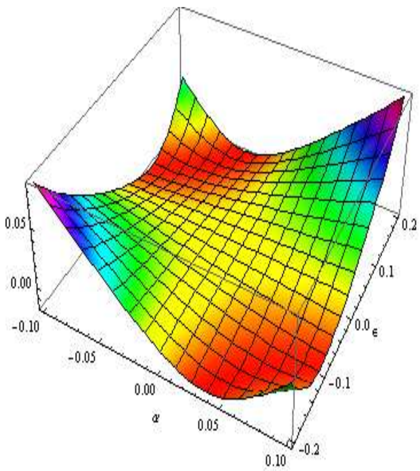
(d1) $\pi_{m,2}^{(f)} - \pi_{m,2}^{(z)}$



(d2) $R_f - \lambda_a^*$



(d3) $\mathbb{E}^{(f)}(\tilde{r}_m) - \mathbb{E}^{(z)}(\tilde{r}_m)$



(d4) $\sigma_m^{(f)} - \sigma_m^{(z)}$

FIGURE 4.4. Impact of the existence of a risk-free security on the market proportion of asset 2, CER, market expected and volatility

In Figure (4.4) and Table (4.2), subscripts (f) is referring to the situation where there exists a risk-free security in the market and subscript (z) is referring the case

$\pi_{m,2}^{(f)} - \pi_{m,2}^{(z)}$	$R_f - \lambda_a^*$	$\mathbb{E}^{(f)}(\tilde{r}_m) - \mathbb{E}^{(z)}(\tilde{r}_m)$	$\sigma_m^{(f)} - \sigma_m^{(z)}$	$\frac{\mathbb{E}^{(f)}(\tilde{r}_m)-r_f}{\sigma_m^{(f)}}$-$\frac{\mathbb{E}^{(z)}(\tilde{r}_m)-(\lambda_a^*-1)}{\sigma_m^{(z)}}$
0.3852	1.69%	1.16%	8.43%	0.0338

TABLE 4.2. Impact of the existence of a risk-free security on the market proportion of asset 2 $\pi_{m,2}^{(f)} - \pi_{m,2}^{(z)}$, CER $R_f - \lambda_a^*$, market expected return $\mathbb{E}^{(f)}(\tilde{r}_m) - \mathbb{E}^{(z)}(\tilde{r}_m)$, volatility $\sigma_m^{(f)} - \sigma_m^{(z)}$ and Sharpe Ratio $\frac{\mathbb{E}(\tilde{r}_m)-r_f}{\sigma_m}$-$\frac{\mathbb{E}(\tilde{r}_m)-(\lambda_a^*-1)}{\sigma_m}$

where no risk-free borrowing or lending are allowed. The numerical results in Table (4.2) show that the existence of the risk-free magnifies the effect of heterogeneity on market equilibrium. Without the risk-free security, the market portfolio consists much less of the riskier asset (asset 2), CER, market expected return and market volatility are all reduced significantly as a result. Most importantly, the Sharpe ratio also decreases without the existence of a risk-free security[3]. Hence adding a risk-free security can benefit the aggregate market in terms of mean-variance efficiency even though it is in net-zero supply.

5. CONCLUSION

Heterogeneity, reflecting diversity and disagreement, among investors in financial markets is very common and it has significant impact on the market. Within the standard mean variance framework, we examine the impact of the heterogeneity, in particular the risk tolerance, optimism/pessisim and confidence/dobut, on the market equilibrium. To make an explicit analysis, we consider a market with two heterogeneous agents, two risky assets and a risk-free asset. By assuming heterogeneity and bounded rationality of investors, we characterize the market equilibrium through a consensus belief and derive the boundedly rational equilibrium. By considering mean-preserved spreads in agents' preference and beliefs in the first and second moments of asset returns, we obtain some analytical results on the impact of the biased belief among the two agents on the market equilibrium, risk premium of the risky assets, market portfolio, and the risk-free rate. The results shed light on the risk premium and risk-free rate puzzles.

The disagreement in this paper is characterized by mean preserved spreads about a benchmark homogeneous belief. It would be interesting to extend the analysis to situations with skewed distribution about the heterogeneous beliefs such as in Abel (2002). Also, the wealth effect is not examined in this paper. In addition, extension to a dynamical model to examine the profitability and survivability of agents with different beliefs would be interesting. We leave these to future research.

[3]In the case without a risk-free security, we use $\lambda_a^* - 1$ which is the expected return of the minimum-variance zero-beta portfolio under the consensus belief in calculating the Sharpe ratio.

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