Application and development of the Fuzzy Analytic Hierarchy Process within a Capital Investment Study

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Abstract

Capital budgeting as a decision process is amongst the most important of all management decisions. The importance (consequents) of the subsequent outcome may bring a level of uncertainty to the judgement making process by the decision maker(s), in the form of doubt, hesitancy and procrastination. This study considers one such problem, namely the choice of type of fleet car to adopt by a small car hire company, a choice that accounts for a large proportion of the company's working capital. With a number of criteria to consider, a fuzzy analytic hierarchy process (FAHP) analysis is undertaken to accommodate for the possible inherent uncertainty. Developments are made to the FAHP method utilised, to consider the preference results with differing levels of imprecision in the pairwise judgements made.

1. Introduction

Capital budgeting is the decision process relating to long-term capital investment programmes. The decisions made are amongst the most important of all management decisions (Smith, 1994). However, numerous capital budgeting decision-making methods only take into account financial criteria, but fail to analyse political and market risks (see Ye and Tiong, 2000). A number of research models take risks into account, but each of them focuses on different factors and has its limitations (*ibid*.)

Two of the major problems that decision-makers (DMs) encounter in making capital budgeting decisions involve both the uncertainty and ambiguity surrounding the different criteria they judge on, including financial and non-financial criteria (Tang, 2003). In the case of investment in new technologies, the problems of uncertainty and ambiguity assume even greater proportions because of the difficulty in estimating the impact of unexpected changes on cash flows (Franz et al., 1995; Sutardi et al., 1995). Moreover, it is difficult to measure the positive impact on cash flows brought about by the increase in quality and quicker reaction to changes in the market (Kaplan, 1986; Franz et al., 1995).

Apart from uncertainty in qualitative (objective) data, the other problem of uncertainty in capital budgeting investment is from DMs' subjective opinions. These uncertainties involve incomplete information, inadequate understanding, and undifferentiated alternatives (Lipshitz and Strauss, 1997). Here we focus on eliciting subjective opinions from DMs, with the ultimate objective, the selection of a best course of action from a set of available alternatives. Moreover, the problem considered here relates to the choice of type of fleet car, to be adopted by a small car hire company, with the choice made by one of its directors. Even for this company, the decision problem affects a large proportion of their working capital. Hence, the importance (consequents) of the decision outcome may bring a level of doubt and hesitancy to the decision making process by the DM (*ibid*.).

To operationalise this decision making process, there exists a number of methods to elicit the DMs' subjective opinions (e.g. CW method in Wang (1997); AHP in Saaty (1980); Java AHP in Zhu and Dale (2001); Randomized Expert Panel Opinion Marginalizing Procedure in Tenekedjiev et al. (2004)). Amongst the most well known is the analytic hierarchy process – AHP (Saaty, 1980). Here to accommodate the acknowledged possible uncertainty (including doubt, hesitancy etc.), in the subjective judgements to be made, a Fuzzy AHP (FAHP) approach is adopted. The earliest work in the FAHP appeared in Laarhaven and Pedryz (1983), which utilised triangular fuzzy numbers to model the pairwise comparisons made in order to elicit weights of preference of the decision alternatives considered. Since then, the FAHP related developments have been consistently reported in the concomitant literature (e.g. the spatial allocation within FAHP (Wu et al., 2004); the method of fuzzy AHP and fuzzy multiple criteria decision making in Hsieh et al. (2004); Mikhailov's (2003) deriving priorities from FAHP; the FAHP revisited within Buckley et al. (2001)).

In this study the synthetic extent method of the FAHP (Chang, 1996; Zhu, 1999; Bozdağ et al., 2003), is further developed and applied to the hire car choice problem discussed previously. Central to this development is the measure of imprecision in the pairwise comparisons made between alternatives (cars), described by triangular membership functions - MFs (e.g. Chiou and Tzeng, 2001; Sohn et al., 2001; Cheng et al., 1999; Deng, 1999; Zhu et al., 1999; Chang, 1996). This imprecision is with respect to the degree of fuzziness in the judgements made (MFs), which is re-defined here from previously used. Further, a sensitivity analysis on the changes in the results preference weights is also given, based on the change in the degree of fuzziness allowed. Throughout this paper, there is a balance between the size of the problem considered and the developments on the existing FAHP technique used, as such there is a level of expositional approach to the application problem investigated.

The structure of the rest of the paper is as follows: In section 2, the details of the hire car choice problem are described. In section 3, the synthetic extent method of the FAHP is presented, including the new developments introduced here. In section 4, the results of the FAHP analysis on the hire car choice problem are illustrated. In section 5, conclusions are given as well as directions for future research.

2. Identification of hire car choice problem

This section presents the details of the capital investment problem investigated throughout this study. The problem concerns a small hire car company and their choice of type of fleet car to be adopted. This choice is an important investment decision, with a large proportion of their working capital to be tied up in their final choice. Here, one of the three directors of the company agreed to make the necessary judgements to be used in a FAHP analysis on this decision problem. However a number of a priori decisions were made on the specific structure of the hire car choice problem.

The first stage was the identification of the necessary criteria to be considered, which here was a consequence of a semi-structured interview with the director (herein defined DM). With the DM told of the expositional nature of the application, it was decided to restrict the number of criteria to only five, through discussion s/he was agreed to be:

i) Equipment: This includes whether the car has central locking, electric windows, power steering, automatic, gear box, air conditioning, and number of doors etc. (see car description table in Appendix A)

ii) Comfort: This includes the space and seating of the cars, and the general décor utilised.

iii) Safety: The most important safety feature is those that reduce the risk of death or serious injury. This criterion includes: airbags, antilock breaking system, impact protection systems, seat belts, safety of the body and number of alarm systems.

iv) Image: The general image of the cars in terms of the market of customer, some emphasis here was simply the colour of the car.

v) Price: This is the price which would be paid for the used car, with also some emphasis on the depreciation of the car, when considered for re-selling.

Apart from the five criteria, the semi-structured interviews also identified five types of cars, which the DM agreed on would be those seriously considered in the car choice problem (see Appendix 1). In summary, the five types of cars are Proton Persona, Honda New Civic, Vauxhall Merit, Volkswagen Polo and Daewoo Lanos, denoted herein A_1, A_2, A_3, A_4 and A_5, respectively.

Given the necessary details of the criteria and decision alternatives, the DM was asked to indicate their preference between pairs of criteria, and then between pairs of alternatives over the different criteria. One characteristic of this questionnaire that is states that if the DM does not have an opinion on any of the comparisons between pairs of criteria and alternatives then s/he should leave the relevant row blank. The questionnaire is designed to avoid pressurising the DM into an inappropriate decision, by allowing for incompleteness.
The linguistic variables used to make the pairwise comparisons, were those associated with the standard 9-unit scale (Saaty, 1980), see Table 1.

Insert Table 1 about here

The use of the linguistic variables is important here, since using the FAHP, the changes in the degree of fuzziness relate to the strength of association with the linguistic terms defined. The results of the pairwise comparisons made by the DM are illustrated in Tables 2 (five criteria) and 3 (five alternatives).

Insert Table2 about here

Insert Table 3 about here

One noticeable aspect of the pairwise comparison tables, are the empty cells, dash lines, these highlight the ability of the DM not to make the exhaustive specific judgements. These judgements are subsequently used to exposit the FAHP analysis, including the development to be introduced.

3. Presentation of Synthetic Extent FAHP method

As a method of multi-criteria decision making, the analytic hierarchy process – AHP (Saaty, 1980) is well established as a versatile technique. It is not surprising that it has been developed to within a fuzzy environment (Zadeh, 1965). The family of fuzzy AHP – FAHP techniques developed all undertake pairwise comparisons, but may or not may adhere to the Saaty based analytic process (see Buckley et al., 2001; Mikhailov, 2003).

In this study the synthetic extent FAHP introduced in Chang (1996) is utilised, which itself was developed in Zhu et al. (1999) and recently applied to the selection of computer integrated manufacturing systems (Bozdağ et al., 2003). One reason for its employment is that from its introduction it allows for the incompleteness of the pairwise judgements made (not the only FAHP approach to allow this, see Interval Probability Theory (Davis and Hall, 2003 also). This offers its suitability in decision problems where uncertainty exists in the judgement making process. A brief exposition of the FAHP method is next given.

3.1 Value of fuzzy synthetic extent

Let $C = \{C_1, C_2, \ldots, C_n\}$ be a criteria set, where n is the number of criteria and $A = \{A_1, A_2, \ldots, A_m\}$ is a DA set where m is the number of DAs. Let $M_{C_i}^1$, $M_{C_i}^2$, …., $M_{C_i}^m$ be values of extent analysis of i^{th} criteria for m DAs. Where $i = 1, 2, \ldots, n$ and all the $M_{C_i}^j$ ($j = 1, 2, \ldots, m$) are triangular fuzzy numbers (TFNs). Then the value of fuzzy synthetic extent (S_i) with respect to the i^{th} criteria is defined as:

$$S_i = \sum_{j=1}^{m} M_{C_i}^j \cdot \left[\sum_{i=1}^{n} \prod_{j=1}^{m} M_{C_i}^j \right]^{-1}, \tag{1}$$

where · represents fuzzy multiplication and superscript –1 represents the fuzzy inverse. The concepts of synthetic extent can also be found in Cheng (1999), Kwiesielewicz (1998) and Bozdağ et al. (2003).

3.2 Construction of the FAHP comparison matrices

The aim of any FAHP method is to elucidate an order of preference on a number of DAs, i.e., a prioritised ranking of DAs. Central to this method is a series of pairwise comparisons, indicating the relative preferences between pairs of DAs in the same hierarchy. It is difficult to map qualitative preferences to point estimates, hence a degree of uncertainty will be associated with some or all pairwise comparison values in an FAHP problem (Yu, 2002). By using triangular fuzzy numbers, via the pairwise comparisons made, the fuzzy comparison matrix $X = (x_{ij})_{n \times m}$ is constructed.

The pairwise comparisons are described by values taken from a pre-defined set of ratio scale values as presented in Table 1. The ratio comparison between the relative preference of elements indexed i and j on a criterion can be modelled through a fuzzy scale value associated with a degree of fuzziness. Then an element of X, x_{ij} (comparison of i^{th} DA with j^{th} DA with respect to a specific criterion) is a fuzzy number defined as $x_{ij} = (l_{ij}, m_{ij}, u_{ij})$, where m_{ij}, u_{ij} and l_{ij} are the modal-value, the upper bound and the lower bound values of a fuzzy number x_{ij}, respectively.

To keep the reciprocal nature of the fuzzy comparison matrix X, the fuzzy number is also satisfied with $l_{ij} = \frac{1}{u_{ji}}$, $m_{ij} = \frac{1}{m_{ji}}$, $u_{ij} = \frac{1}{l_{ji}}$. More formally, given an element x_{ij} in the fuzzy comparison matrix has modal-value scale value v_k. For instance, if there is a strong preference[1] of an element i over an element j under a certain criterion: then $x_{ij} = (l_{ij}, 5, u_{ij})$ and $v_k = 5$, with the l_{ij} and u_{ij} values found depending on the associated degree of fuzziness.

3.3 Calculation of the sets of weight values of the FAHP

To obtain the estimates for the sets of weight values under each criterion, it is necessary to consider a principle of comparison for fuzzy numbers (Chang, 1996). For example, for two fuzzy numbers M_1 and M_2, the degree of possibility of $M_1 \geq M_2$ is defined as:

$$V(M_1 \geq M_2) = \sup_{x \geq y} [\min(\mu_{M_1}(x), \mu_{M_2}(y))].$$

Where sup represents supremum (i.e., the least upper bound of a set) and when a pair (x, y) exists such that $x \geq y$ and $\mu_{M_1}(x) = \mu_{M_2}(y) = 1$, then it follows that $V(M_1 \geq M_2) = 1$ and $V(M_2 \geq M_1) = 0$. Since M_1 and M_2 are convex fuzzy numbers defined by the TFNs (l_1, m_1, u_1) and (l_2, m_2, u_2) respectively, then

$$V(M_1 \geq M_2) = 1 \text{ iff } m_1 \geq m_2; \qquad V(M_2 \geq M_1) = \text{hgt}(M_1 \cap M_2) = \mu_{M_1}(x_d). \tag{2}$$

Where iff represents "if and only if" and d is the ordinate of the highest intersection point between the μ_{M_1} and μ_{M_2} TFNs (see Figure 1) and x_d is the point on the domain of μ_{M_1} and μ_{M_2} where the ordinate d is found. The term hgt is the height of fuzzy numbers on the intersection of M_1 and M_2. For $M_1 = (l_1, m_1, u_1)$ and $M_2 = (l_2, m_2, u_2)$, the possible ordinate of their intersection is given by the expression (2). The degree of possibility for a convex fuzzy number can be obtained from the use of equation (3).

$$V(M_2 \geq M_1) = \text{hgt}(M_1 \cap M_2) = \frac{l_1 - u_2}{(m_2 - u_2) - (m_1 - l_1)} = d. \tag{3}$$

Insert Figure 1 about here

One point of concern, highlighted in this paper is when two elements (fuzzy numbers - M_1 and M_2) say (l_1, m_1, u_1) and (l_2, m_2, u_2) in a fuzzy comparison matrix satisfy $l_1 - u_2 > 0$ (see Figure 2) then $V(M_2 \geq M_1) = \text{hgt}(M_1 \cap M_2) = \mu_{M_2}(x_d)$, with $\mu_{M_2}(x_d)$ given by (Zhu et al., 1999):

$$\mu_{M_2}(x_d) = \begin{cases} \dfrac{l_1 - u_2}{(m_2 - u_2) - (m_1 - l_1)} & l_1 \leq u_2; \\ 0 & \text{others.} \end{cases} \tag{4}$$

Insert Figure 2 about here

[1] The original mapping from a linguistic scale to numerical values (ratios) is referred to in Table 1.

The degree of possibility for a convex fuzzy number M to be greater than the number of k convex fuzzy numbers M_i (i = 1, 2, …, k) can be given by the use of the operation max and min (Dubois and Prade, 1980) and can be defined by:

$$V(M \geq M_1, M_2, \ldots, M_k) = V[(M \geq M_1) \text{ and } (M \geq M_2) \text{ and } \ldots \text{ and } (M \geq M_k)] = \min V(M \geq M_i),\ i = 1, 2, \ldots, k.$$

Assume that $d'(A_i) = \min V(S_i \geq S_k)$, where k = 1, 2, …, n; $k \neq i$ and n is the number of criteria as described previously. Then a weight vector is given by:

$$W' = (d'(A_1), d'(A_2), \ldots, d'(A_m)). \quad (5)$$

where A_i (i = 1, 2, …, m) are the m DAs. Hence each $d'(A_i)$ value represents the relative preference of each DA. To allow the values in the vector to be analogous to weights defined from the AHP type methods then the vector W' is normalised and denoted by:

$$W = (d(A_1), d(A_2), \ldots, d(A_m)). \quad (6)$$

Referring back to fuzzy numbers, for example an element x_{ij} in a fuzzy comparison matrix, if DA i is preferred to DA j then m_{ij} takes an integer value from two to nine (from the 1-9 scale). More formally, given the entry m_{ij} in the fuzzy comparison matrix has the k^{th} scale value v_k, then l_{ij} and u_{ij} have values either side of the v_k scale value. It follows, the values l_{ij} and u_{ij} directly describe the fuzziness of the judgement given in x_{ij}. In Zhu et al. (1999) this fuzziness is influenced by a δ (degree of fuzziness) value, where $m_{ij} - l_{ij} = u_{ij} - m_{ij} = \delta$. That is, the value of δ is a constant and is considered an absolute distance from the lower bound value (l_{ij}) to the modal value (m_{ij}) or the modal value (m_{ij}) to the upper bound value (u_{ij}), see Figure 3.

Insert Figure 3 about here

Given the modal value (scale value) m_{ij} (v_k), the fuzzy number representing the fuzzy judgement made is defined by ($m_{ij} - \delta$, m_{ij}, $m_{ij} + \delta$), with its associated inverse fuzzy number subsequently described by $\left(\frac{1}{m_{ij}+\delta}, \frac{1}{m_{ij}}, \frac{1}{m_{ij}-\delta}\right)$.

In Figure 3 the definition of the fuzzy scale value given in Zhu et al. (1999) is that the distance from m_{ij} (= v_k) to v_{k-1} is equal to the distance from m_{ij} to v_{k+1}, which implies the distances from m_{ij} to l_{ij} and m_{ij} to u_{ij} are equal to each other (δ distance). In the case of m_{ij} given a value of one (m_{ij} = 1) off the leading diagonal ($i \neq j$), the general form of its associated fuzzy scale value is defined as $\left(\frac{1}{1+\delta}, 1, 1+\delta\right)$.[2] For example, given m_{ij} = 1, the fuzzy number will be (0.6667, 1, 1.5) when δ = 0.5.

When considering the domain of δ, if δ = 0 then $l_{ij} = m_{ij} = u_{ij}$ hence a non-fuzzy (crisp) number. As δ increases in size, so the distance value $u_{ij} - l_{ij}$ increases, and greater is the inherent degree of fuzziness in the judgements. On the issue of the δ value, Zhu et al. (1999) state:

> " … *when* $0 < \delta < 0.5$, m_{ij} *is selected as the consecutive two-level scale. At this time* $\mu(d)$ [$\mu_{M_2}(x_d)$] = 0 [see Figure 2], *but it doesn't reflect the cognitive fuzziness completely. When* $\delta > 1$, *the degree of fuzziness increases and the degree of confidence decreases. The practical result indicates that* $0.5 \leq \delta \leq 1$ *is more suitable*. "

This debate on the value of δ and its associated workable (suitable) domain will be discussed in Section 4.

There are two factors to consider here regarding the δ expression, firstly in Zhu et al. (1999) the value δ is an absolute value. That is, it defines the physical distance the l_{ij} and u_{ij} values are away from the modal value m_{ij}. For example, when δ = 0.5 then $l_{ij} = m_{ij} - 0.5$ and $u_{ij} = m_{ij} + 0.5$. The second factor leads to a better understanding of the role of the δ value on the support associated with each fuzzy preference judgement. One restriction of the method described by Zhu et al. (1999) is that it assumes equal unit distances between successive scale values. However with respect to the traditional AHP there has been a growing debate on the actual appropriateness of the Saaty 1-9 scale, with a number of alternative sets of scales being proposed (see Beynon (2002) and Tang (2003) and references contained therein).

These (possible) alternative sets of scales do not necessarily have equal distances between their successive scale values (see Figure 4). To allow the possibility of utilising scale values with non-equal inter distances between them, the definition of δ is redefined. Here, δ is defined as a proportion (relative) of the distance between successive scale values. Hence the associated fuzzy scale value for the case of m_{ij} given scale value v_k is defined as:

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[2] The expression for $\left(\frac{1}{1+\delta}, 1, 1+\delta\right)$ is supported by Escobar and Moreno-Jiménez (2000) who for the traditional AHP state that the distribution of the scale value above and below one are analogous.

$$(v_k - \delta\,(v_k - v_{k-1}),\ v_k,\ v_k + \delta\,(v_{k+1} - v_k)). \qquad (7)$$

Therefore, $m_{ij} = v_k$, $l_{ij} = v_k - \delta\,(v_k - v_{k-1})$ and $u_{ij} = v_k + \delta\,(v_{k+1} - v_k)$. When the maximum scale value v_9 is used, consideration has to be given to its associated upper bound values. That is given $m_{ij} = v_k$ then it is not possible to use the previously defined expressed, instead of $u_{ij} = u_9 = v_9 + \delta \frac{(v_9 - v_8)^2}{(v_8 - v_7)}$. The reason is that there is no v_{10} (v_{9+1}) value to use, so instead the new expression takes into account the difference between successive scale values. That is, using the notation v_{10} value we would require: $\frac{(v_{10} - v_9)}{(v_9 - v_8)} = \frac{(v_9 - v_8)}{(v_8 - v_7)}$; which becomes $(v_{10} - v_9) = \frac{(v_9 - v_8)^2}{(v_8 - v_7)}$. Hence our expression for u_9 is $v_9 + \delta \frac{(v_9 - v_8)^2}{(v_8 - v_7)}$. Bringing in the 1-9 scale, when $v_9 = 9$ then $u_9 = 9 + \delta \frac{(9-8)^2}{(8-7)} = 9 + \delta$. That is, the upper bound value uses information from the inter-distance between the two preceding scale values. A graphical representation where δ is a proportional (relative) value is provided in Figure 4.

Insert Figure 4 about here

In Figure 4, the diagram shows the case when distances between successive scale values are not equal ($v_k - v_{k-1} \neq v_{k+1} - v_k$). Here the effect of the δ value on a fuzzy number (l_{ij}, m_{ij}, u_{ij}) will be elucidated. For example, around this scale value 1, the domain of the fuzzy scale value measure is between 0 and ∞.[3] With the sub-domain 0 to 1 associated with one direction of preference (e.g., *j* preferred to *i*) and 1 to ∞ the reverse preference (e.g., *i* preferred to *j*). In the case of fuzzy scale values, there is still a need for the strict partition of the scale value domain. That is, the support of any fuzzy scale value should be in either the 0 to 1 or the 1 to ∞ sub-domains of δ.

If this were not satisfied then there would be sub-domains of the support which would be in conflict with each other.[4] This approach follows a direction of study in the traditional AHP, when interval preference judgements are utilised. That is, the partition of the domain of the scale value to the sub-domains 0 to 1 and 1 to ∞ is adhered to over the domain of the interval judgements around a scale value (see Moreno-Jiménez and Vargas, 1993; Bryson and Joseph, 2000).

Insert Figure 5 about here

To illustrate, using the fuzzy scale value $m_{ij} = v_k = 2$, following Zhu et al. (1999) if $\delta = 1.5$ the associated fuzzy number is (0.5, 2, 3.5) (see Figure 5). It follows, $l_{ij} = 0.5 < 1$ and implies that a sub-domain of the support (0.5, 1) is meaningless with the fuzzy scale value $m_{ij} = 2$. One further example shows that when $\delta = 2.5$, the associated fuzzy number is (–0.5, 2, 4.5), and the value –0.5 has no meaning as part of a fuzzy judgement (ratio scale measure). To remove this potential of conflict, a restraint on the l_{ij} value needs to be constructed.[5] Expressed more formally, if m_{ij} is given a fuzzy scale value such that $m_{ij} = v_k \geq 1$ then l_{ij} is bound by $1 \leq l_{ij} \leq m_{ij}$, whose value depends on the value of δ, and is given by:

$$l_{ij} = \begin{cases} v_k - \delta(v_k - v_{k-1}) & \delta \leq \dfrac{v_k - 1}{v_k - v_{k-1}}; \\ 1 & \delta > \dfrac{v_k - 1}{v_k - v_{k-1}}. \end{cases} \qquad (8)$$

This expression for l_{ij} ensures that irrespective of the value of δ, the support associated with a fuzzy scale value includes no conflicting sub-domain. There is no limit on the upper bound of the fuzzy scale value, hence the value of u_{ij} remains as $u_{ij} = v_k + \delta(v_{k+1} - v_k)$. A similar discussion and subsequent expression can be given for the inverse case (when $m_{ij} < 1$). To further illustrate this expression for the l_{ij} value, Figure 6 shows the respective graphical presentation of the fuzzy number characterisation of a fuzzy scale value for different values (sub-domain) of δ. The fuzzy scale value considered is $v_k = 3$, so the associated fuzzy number has modal value $m_{ij} = 3$. It is presumed that neighbour scale values are a unit distance either side of the scale value 3, hence $v_{k-1} = 2$ and $v_{k+1} = 4$.

Insert Figure 6 about here

In Figure 6, the graphical representation of the fuzzy number (l_{ij}, 3, u_{ij}) is demonstrated, dependent on the δ value. That is, the differently shaded regions define where the left and right bounds of the support for the TFN will lie within, for certain sub-domains of δ. Since there is a unit distance between successive scale

[3] Considering the upper bound of ∞, implies an unbounded scale, see Jensen (1984) for a discussion on the notion of an unbounded scale. It is stressed the notion of ∞ is with the domain of support rather than the actual modal value scale, taken from the Saaty 1-9 scale here.

[4] The exception is when $m_{ij} = 1$ then the fuzzy scale value has support which spans either side of the scale value 1.

[5] This argument assumes that the fuzziness in a preference judgement ($m_{ij} \neq 1$) does not include the possibility of reversal in the direction of the preference initially implied.

values, the majority of the results are symmetrical around the special case of $l_{ij} = m_{ij} = u_{ij} = 3$ (crisp value). The darkest shaded regions (either side of the $m_{ij} = 3$ value) in Figure 6 are the domains of the l_{ij} and u_{ij} values for when δ is between (0, 0.5], which implies $2.5 \leq l_{ij} < 3$ and $3 < u_{ij} \leq 3.5$.

The middle shaded region is when δ is in (0.5, 1], which implies $2 \leq l_{ij} < 2.5$ and $3.5 < u_{ij} \leq 4$, following the Zhu et al. (1999) statement, it is suggested the region is considered the most suitable δ domain. The lightly shaded region when δ is in (1, 2], implies $1 \leq l_{ij} < 2$ and $4 < u_{ij} \leq 5$. In the case of $\delta > 2$ it is necessary to refer specifically to equation (8) for l_{ij}. That is, the lower bound on l_{ij} when $\delta = 2$ and shown by the light shaded region in Figure 6 is the value 1.

This is the least value the lower bound can take, so from equation (8) the value of l_{ij} for when $\delta > 2$ is always 1. To illustrate, when δ in (2, 4], then $l_{ij} = 1$ and $5 < u_{ij} \leq 7$, as shown with the non-shaded region in Figure 6. In this case there is no equivalent non-shaded region in Figure 6 representing the range of the l_{ij} value for δ in (2, 4], since it is always $l_{ij} = 1$.

Following the exposition of the FAHP extent analysis as stated above, the subjective opinions in Tables 2 and 3 need to be transferred into the fuzzy comparison matrix. An example demonstrates the transformation of Table 2 as shown in Table 4.

Insert Table 4 about here

In Table 4 the 1-9 scale is utilized, therefore, the distances between successive scale values are equal ($v_k - v_{k-1} = v_{k+1} - v_k$). Here in this case, the effect of the fuzzy numbers is only the changing of δ value.

3.4 The advantages of this method

The utilisation of the FAHP extent analysis method presented in this paper brings together a number of advantageous aspects of group decision-making in a fuzzy environment. That is, while many of these aspects are present in other techniques, they are most present in this FAHP extent analysis method, and will now be discussed.

Group decision-making

The role of group decision-making is increasingly important (Ahn, 2000). The method used in this paper takes account of group decision-making (see equation (1)). Each matrix can involve all the DMs' judgements. However, in this case study only one of the directors from the car hire company is willing to answer the questionnaire.

Computational manageability

The computational manageability of the FAHP method in this paper allows results in the form of weight values to be evaluated in a small amount of time. It does not require the evaluation of fuzzy eigenvalues (Juang and Lee, 1991) or the solving of a linear programming problem (e.g., in Arbel, 1989; Bryson and Joseph, 1999; Yu, 2002), instead it utilises a number of rows and columns averaged from the associated fuzzy comparison matrices (Beynon and Tang, 2002; Tang, 2003). It also allows the opportunity for a level of sensitivity analysis to be realistically undertaken on the comparison matrices (*ibid.*). In this paper the sensitivity analysis undertaken (see later) relates to the change in the fuzziness degree associated with the preferences (judgements) made.

Imprecision

A major consequence of the incorporation of decision-making in a fuzzy environment is the acknowledgement of and allowance for imprecision in the judgements made. Imprecision refers to the contents of the considered judgements and depends on the "granularity" of the language used in those judgements (Bosc and Prade, 1997). The method in this paper allows the judgements in the judgement matrices to be given a measure of imprecision by using the degree of fuzziness - δ (the quantifiable allowance for a level of imprecision in the judgement(s) made).

Incompleteness

One aspect of the FAHP method of this paper is the prevalence of and allowance for incompleteness in the judgements made by DMs. For example, if a DM is not willing or able to specify the preference judgements in the detailed way required by the corresponding method then, a DM is able to not make a judgement in the form of a pairwise comparison between two DAs. The problem of DMs being unable to provide complete information in the above circumstances is addressed by the allowance for incompleteness in the FAHP

4. Results of the FAHP analysis in the hire car problem

In this section, the concepts presented above are applied to the data from the hire car selection case study. The redefinition of the proportional distance between lower bound and upper bound values associated with fuzzy numbers in the FAHP in Section 3 is now applied in a practical environment, to reach a decision on capital investment. The application of the FAHP to the data from the hire car selection case study is described as follows.

4.1 The process of weight evaluation

Utilizing the expression given in Section 3, we apply the FAHP extent analysis method to the data on capital budgeting case study previously described. The following stages demonstrate how to obtain the weight values for DAs. In this demonstration, the degree of fuzziness is set up at 0.5. [6]

Weights evaluation for criteria

In this car selection case study, only the judgements between criteria obtained from the DM will be demonstrated. Subsequently, the judgements between DAs over different criteria are dealt with in an identical manner. The calculation of the fuzziness degree within each scale value will be transformed from Table 4 to Table 5. The first stage of the weight evaluation process is the aggregation of l_{ij}, m_{ij} and u_{ij} values, present in the pairwise comparison matrix for the judgements between criteria. Following the fuzzy synthetic extent concept shown in equation (1), the evaluation with respect to five criteria in terms of the 1-9 scale from Saaty (1980) based on $\delta = 0.5$ can be illustrated as shown in Table 6:

Insert Table 6 about here

The associated S_i values can be found as follows:

$S_1 = (6.1818, 7.2000, 8.2222) \cdot (\frac{1}{47.1826}, \frac{1}{42.8207}, \frac{1}{38.4128})$
$= (0.1310, 0.1681, 0.2140)$;

$S_2 = (1.7968, 1.9444, 2.1843) \cdot (\frac{1}{47.1826}, \frac{1}{42.8207}, \frac{1}{38.4128})$
$= (0.0381, 0.0454, 0.0569)$;

$S_3 = (16.6667, 18.000, 19.000) \cdot (\frac{1}{47.1826}, \frac{1}{42.8207}, \frac{1}{38.4128}) = (0.3532, 0.4204, 0.4946)$;

$S_4 = (1.6001, 1.6762, 1.7761) \cdot (\frac{1}{47.1826}, \frac{1}{42.8207}, \frac{1}{38.4128}) = (0.0339, 0.0391, 0.0462)$;

$S_5 = (12.1667, 14.000, 16.000) \cdot (\frac{1}{47.1826}, \frac{1}{42.8207}, \frac{1}{38.4128}) = (0.2579, 0.3269, 0.4165)$;

Using equations (2) and (3) described in Section 3

$V(S_1 \geq S_2) = 1$; $V(S_1 \geq S_3) = \frac{0.3532 - 0.2140}{(0.1681 - 0.2140) - (0.4204 - 0.3532)} = -1.2308 = 0$;

$V(S_1 \geq S_4) = 1$; $V(S_1 \geq S_5) = 0$; $V(S_2 \geq S_1) = 0$;

$V(S_2 \geq S_3) = 0$; $V(S_2 \geq S_4) = 1$; $V(S_2 \geq S_5) = 0$;

$V(S_3 \geq S_1) = 1$; $V(S_3 \geq S_2) = 1$; $V(S_3 \geq S_4) = 1$; $V(S_3 \geq S_5) = 1$; $V(S_4 \geq S_1) = 0$; $V(S_4 \geq S_2) = \frac{0.0381 - 0.0462}{(0.0391 - 0.0462) - (0.0454 - 0.0381)} = 0.5655$;

$V(S_4 \geq S_3) = 0$; $V(S_4 \geq S_5) = 0$; $V(S_5 \geq S_1) = 1$; $V(S_5 \geq S_2) = 1$; $V(S_5 \geq S_3) = \frac{0.3532 - 0.4165}{(0.3269 - 0.4165) - (0.4204 - 0.3532)} = 0.4039$;

$V(S_5 \geq S_4) = 1$.

Finally, using equation (4) described in Section 3, it follows that

$d'(C_1) = V(S_1 \geq S_2, S_3, S_4, S_5) = \min(1, 0, 1, 0) = 0$,

$d'(C_2) = V(S_2 \geq S_1, S_3, S_4, S_5) = \min(0, 0, 1, 0) = 0$,

$d'(C_3) = V(S_3 \geq S_1, S_2, S_4, S_5) = \min(1, 1, 1, 1) = 1$,

$d'(C_4) = V(S_4 \geq S_1, S_2, S_3, S_5) = \min(0, 0.5655, 0, 0) = 0$,

$d'(C_5) = V(S_5 \geq S_1, S_2, S_3, S_4) = \min(1, 1, 0.4039, 1) = 0.4039$.

Therefore,

$$W' = (0, 0, 1, 0, 0.4039).$$

Through normalization, the weight vectors are obtained with respect to the decision criteria C_1, C_2, C_3, C_4 and C_5:

$$W = (0, 0, 0.7123, 0, 0.2877).$$

Similarly, the transformation procedures for comparisons between criteria based on other alternative scales can be found and the final results based on $\delta = 0.5$ are shown below.

Insert Table 7 about here

Table 7 shows that A_4 has the largest weight while $\delta = 0.5$. It reveals that DM prefers the Volkswagen car than others. The next is Daewoo. However, it is not known the preferences for A_1, A_2 and A_3 obtained from Table 7 since there is no weight with these three DAs while $\delta = 0.5$.

The results from Table 7 cannot fully represent the preferences for criteria. That is, since pairwise

[6] The degree of fuzziness is not necessary to be 0.5. It can be any numbers (explain later).

comparisons are made between criteria (or DAs) then it is expected that all weights should have positive values. Zahir (1999) discussed this aspect within the traditional AHP, suggesting that the DM does not favour one criterion (or DA) and ignore all others, rather places the criteria (or DAs) at various levels. Furthermore it is suggested the 1-9 scale forces the concentration of the weight values, whereas only with an unbounded scale range would it be possible for the weights to overwhelmingly prefer on criterion (or DA).

Besides, in accordance with the aspects of Oskamp (1982) that people become confident enough to make decision (i.e., people who don't know how much they don't know). Hence the degree of fuzziness should be enlarged beyond 0.5 rather than only within a certain domains. Therefore, there has an attempt by using sensitivity analysis to observe the weight values changing on different criteria and discuss below.

4.2 Sensitivity analysis of resultant weight values

The objective of a typical sensitivity analysis is to find out when the input data (preference judgements and degrees of fuzziness) are changed into new values, how the ranking of the DAs will change. To illustrate in this paper, firstly we consider the judgements made between the criteria based on the DM, exposited in Tables 2.

Insert Figure 7 about here

There are five lines in Figure 7 which represent the weight values associated with the different criteria. In Figure 7 the numbers (with criteria) on the δ-axis represent the degree of fuzziness appearance points with respect to each criterion. For example the degree of fuzziness δ up to 0.3 (on Figure 7 δ-axis), shows that C_3 has the absolute dominant preference. This means that Safety is an important criterion to be considered when the DM makes decisions and the weight value is 1. After δ reaches 0.3, the criterion C_5 has weight. The next criterion is C_1 which has weight as δ approaches 1.14, etc. The values of δ at which the criteria (or DAs) have positive weight values (non-zero) are hereafter referred to as appearance points.

There is one cross points, of C_2 and C_4 where $\delta = 3.85$. C_4 has greater preference over C_2 after $\delta > 3.85$. Hence the ranking orders become clear after $\delta > 3.85$. The final ranking order shown on the right hand of Figure 7 is C_3, C_5, C_1, C_4, C_2.

For the fuzzy comparison matrix with judgements between criteria (see Figure 7), all the five criteria have positive weights when δ is greater than 3.5. This means that if δ is less than 3.5 some criteria have no positive weights. It is suggested therefore that it is useful to choose a minimum workable degree of fuzziness. The expression minimum workable degree of fuzziness is defined as the largest of the values of δ at the various appearance points of criteria (or DAs) on the δ-axis. For example in Figure 7, the weights of the criteria C_5, C_1, C_2, and C_4 at the appearance points are 0.3, 1.14, 3.36, and 3.5, respectively. The largest of these appearance points is 3.5. Hence for this fuzzy comparison matrix a minimum workable δ value can be expressed as δ_{T_C} = 3.5 where the subscript TC represents the comparisons between different criteria.

Figure 7 can also be compared with Table 2. In Table 2, the Safety C_3 has most of the preference when comparing it with other criteria apart from has no opinion on comparing with C_1. The Price C_5 also has the most of the preferences when compared with other criteria. The least preference shown in Figure 7 (where $\delta < 3.85$) is Image C_4. This also can be verified from Table 2. In the C_4 row shown in Table 2, there is no preference made by the DM between criteria. It seems as if the Image (the colour) of the DA is not very important when DM is making their decisions. Although the order changes between C_2 and C_4 after $\delta > 3.85$, they both are still very close to each other as shown in Figure 7.

In Figure 7 the degree of fuzziness within domains 0 to 1 only two criteria have positive weights. The results are against the extant research (i.e., Zhu et al., 1999), δ should be within the domain 0 to 1.

Referring the comparisons between DAs on these five criteria, Figures 8*a* to *e* shows the varied movement of the weight values as δ within domain 0 to 5. For the fuzzy comparison matrix with judgements between DAs on different criteria (see Figures 8*a* to *e*), all the DAs have positive weights when δ is greater than 4.73 (see Figure 8*c*). This means that if δ is less than 4.73 some DAs have no positive weights.

Insert Figure 8 about here

For the comparisons between DAs with respect to individual criterion fuzzy comparison matrices (on C_1, C_2, C_3, C_4, C_5, C_6 and C_7) their minimum workable δ values are $\delta_{T_1} = 2.75$, $\delta_{T_2} = 4.55$, $\delta_{T_3} = 4.73$, $\delta_{T_4} = 4.55$, and $\delta_{T_5} = 2.86$, respectively (see Figure 8*a* to *e*).

When considering the final results, the domain of workable δ is expressed as δ_T, and is defined by the maximum of the various minimum workable degrees of fuzziness throughout the problem, that is here $\delta_T = \max(\delta_{T_C}, \delta_{T_1}, \delta_{T_2}, \delta_{T_3}, \delta_{T_4}, \delta_{T_5}, \delta_{T_6}, \delta_{T_7}) = 4.73$ where the subscript T is the maximum of the minimum workable δ values in the six fuzzy comparison matrices. It follows that for this problem the workable region of

δ is $\delta > 4.73$ and the results on weights should possibly only be considered in the workable δ region. The use of minimum workable degree of fuzziness is intended to exclude values of δ at which there are no positive weights for the DAs. However the use of a workable value of δ is not to be strictly enforced.

Insert Table 6 about here

In Table 8, the final results show that the most preferred car is the Honda (A_2), and then the Volkswagen (A_4), the Vauxhall (A_3), the Daewoo (A_5) and the Proton (A_1), which is the least preferred car in the DM's mind. From the comparison between the criteria in Table 8, the first two preferred criteria out of five criteria are Safety and Price. It means that the DM cares about the cost (car price) and safety more than other criteria.

In this study, the phenomenon of rank reversal happens when sets of weight values are obtained in the pairwise comparisons between criteria (see Figure 7). Each fuzzy comparison matrix contains uncertain and incomplete information. Therefore when these fuzzy comparison matrices are aggregated together they also involve imperfect information. The increase in degree of fuzziness also increases the uncertainty of the information.

One thing which should be highlighted here is that the use of the minimum workable degree of fuzziness to obtain the final aggregation results does not take into account the possibility that the phenomenon of rank reversal might exist when δ becomes larger. This shows the need for more research in future work.

5. Conclusions

The aim of this study is to investigate the application of the Fuzzy Analytic Hierarchy Process (FAHP) method of multi-criteria decision making (MCDM), within a capital budgeting problem. The application problem in question is the choice of type of fleet car to be adopted by a small car hire company. The important consequences of the choice outcome may confer a level of uncertainty on the decision maker, in the form of doubt, procrastination etc. This is one reason for the utilisation of FAHP, with its allowance for imprecision in the judgements made. The issue of imprecision is reformulated in this study, which further allows a sensitivity analysis on the preferences weights evaluated to changes in the levels of imprecision.

It is found the DM (one director) of the car hire company successfully made the necessary judgements made. This included their allowance to not make specific pairwise comparisons between all pairs of decision alternatives – the incompleteness another aspect of the possible inherent uncertainty in the decision process. The re-definement of the degree of fuzziness associated with the preference judgements made allows the change of imprecision (fuzziness) to be succinctly reported.

The future research associated with FAHP includes, from the MCDM point of view, those developments with the traditional AHP. These include the appropriateness of the 9-unit scale (integer values one to nine), which within AHP is still an ongoing issue. The effect of using different 9-unit scale within FAHP would further elucidate the sensitivity analysis issues. The graphical results presented in the paper including changes in the degree of fuzziness would clearly exposit this.

Amongst the criteria in the hire car selection problem was price, from its definition this has an associated value with each alternative, hence is a tangible criterion. Within AHP and subsequently FAHP, an ongoing question is how to effectively incorporate the tangible with intangible criteria. Specifically to FAHP, whether the change in the degree of fuzziness may aid in this appropriateness, is again left for future research. An important development in this study is the notion of a workable degree of fuzziness, possibly specific to the synthetic extent FAHP, it needs adoption in future studies to strengthen its appropriateness.

Reference:

Ahn, B.S., The analytic hierarchy process in an uncertain environment: A simulation approach by Hauser and Tadikamalla (1996), *European Journal of Operational Research* **124(1)**, 2000, 217-218.

Arbel, A., Approximate articulation of preference and priority derivation, *European Journal of Operational Research* **43(3)**, 1989, 317-326.

Beynon, M.J., An analysis of distributions of priority values from alternative comparison scales within AHP, *European Journal of Operational Research* **140(1)**, 2002, 104-117.

Beynon, M.J., Tang, Y.C., A further investigation and exposition of a fuzzy AHP utilising the extent analysis method, Working paper, Cardiff University Business School, UK, 2002.

Bosc, P., Prade, H., An introduction to the fuzzy set and possibility theory-based treatment of flexible queries and uncertain or imprecise databases, In: Motro, A., Smets, P., *Uncertainty Management in Information Systems from Needs to Solutions*, Kluwer Academic Publishers, Brussels Belgium, 1997, 285-324.

Bozdağ, C.E., Kahraman, C., Ruan, D., Fuzzy group decision making for selection among computer integrated manufacturing systems, *Computers in Industry* **51**, 2003, 13-29.

Bryson, N., Joseph, A., Generating consensus priority interval vectors for group decision-making in the AHP, *Journal of Multi-Criteria Decision Analysis* **9**, 2000, 127-137.

Bryson, N., Joseph, A., Generating consensus priority vectors: a logarithmic goal programming approach, *Computers & Operations Research* **26**, 1999, 637-643.

Buckley, J.J., Feuring T., Hayashi, Y., Fuzzy hierarchical analysis revisited, *European Journal of Operational Research* **129(1)**, 2001, 48-64.

Chang, D.Y., Applications of the extent analysis method on fuzzy AHP, *European Journal of Operational Research* **95(3)**, 1996, 649-655.

Cheng, C.H., Evaluating weapon systems using ranking fuzzy numbers, *Fuzzy Sets and Systems* **107**, 1999, 25-35.

Cheng, C.H., Lin, Y., Evaluating the best main battle tank using fuzzy decision theory with linguistic criteria evaluation, *European Journal of Operational Research* **142(1)**, 2002, 174-186.

Cheng, C.H., Yang, K.L., Hwang, C.L., Evaluating attack helicopters by AHP based on linguistic variable weight, *European Journal of Operational Research* **116(2)**, 1999, 423-435.

Chiou, H.K., Tzeng, G.H., Fuzzy hierarchical evaluation with grey relation model of green engineering for industry, *International of Fuzzy System* **3(3)**, 2001, 466-475.

Davis, J.P., Hall, J.W., A software-supported process for assembling evidence and handling uncertainty in decision-making, *Decision Support Systems* **35**, 2003, 415-433.

Deng, H., Multicriteria analysis with fuzzy pairwise comparison, *International Journal of Approximate Reasoning* **21**, 1999, 215-231.

Dubois, D., Prade, H., *Fundamentals of Fuzzy Sets-The Handbooks of Fuzzy Sets Series*, Kluwer Academic Publishers, 2000.

Escobar, M.T., Moreno-Jiménez, J.M., Reciprocal distributions in the analytic hierarchy process, *European Journal of Operational Research* **123(1)**, 2000, 154-174.

Franz, D.P., Duke, J.C., Omer, K., Capital budgeting: development of a decision approach for determining expected value of cash flows, *Studies in Managerial and Financial Accounting* **3**, 1995, 199-211.

Hsieh, T.Y., Lu, S.T., Tzeng, G.H., Fuzzy MCDM approach for planning and design tenders selection in public office buildings, *International Journal of Project Management*, 2004, xxx.

Jensen, R.E., An alternative scaling method for priorities in hierarchical structures, *Journal of Mathematical Psychology* **28**, 1984, 317-332.

Juang, C.H., Lee, D.H., A fuzzy scale for measuring weights of criteria in hierarchical structures, *International Fuzzy Engineering Symposium* **91**, 1991, 415-421.

Kaplan, R.S., Must CIM be justified by faith alone?, Harvard Business Review*, 1986, 87-93.*

Kwiesielewicz, M., A note on the fuzzy extension of Saaty's priority theory, *Fuzzy Sets and Systems* **95**, 1998, 161-172.

Li, Q.D., The random set and the cutting of random fuzzy sets, *Fuzzy Sets and Systems* **86**, 1997, 223-234.

Lipshitz, R., Strauss, O., Coping with uncertainty: A naturalistic decision-making analysis, *Organizational Behavior and Human Decision Processes* **69(2)**, 1997, 149-163.

Mikhailov, L., Deriving priorities from fuzzy pairwise comparison judgements, *Fuzzy Sets and Systems* **134**, 2003, 365-385.

Moreno-Jiménez, J.M., Vargas, L.G., A probabilistic study of preference structures in the analytic hierarchy process with interval judgements, *Mathematical and Computer Modeling* **17(4/5)**, 1993, 73-81.

Oskamp, S., Overconfidence in case-study judgements, in Dahneman, D., Slovic, P., Tversky, A., *Judgment under uncertainty:*

Heuristics and biases, Cambridge University Press, 1982.
Saaty, T.L., *The Analytical Hierarchy Process*, McGraw Hill, New York, 1980.
Smith, D.J., Incorporating risk into capital budgeting decisions using simulation, *Management Decision* **32(9)**, 1994, 20-26.
Sohn, K.Y., Yang, J.W., Kang, C.S., Assimilation of public opinions in nuclear decision-making using risk perception, *Energy* **28**, 2001, 553-563.
Sutardi, C.R., Bector, Goulter, I., Multi-objective water resources investment planning under budgetary uncertainty and fuzzy environment, *European Journal of Operational Research* **82(3)**, 1995, 556-591.
Tang, Y.C., *Developing the Application of the Fuzzy Analytic Hierarchy Process to Group Decision-Making: A Capital Investment Case Study*, Thesis, Cardiff University Business School, UK, 2003.
Tenekedjiev, K., Kamenova, S., Nikolova, N., Formalized procedure for ranking alternatives in developing environmental programs, *Journal of Cleaner Production* **12**, 2004, 353-360.
van Laarhoven, P.J.M., Pedrycz, W., A fuzzy extension of Saaty's priority theory, *Fuzzy Sets and Systems* **11**, 1983, 229-241.
Wang, W., Subjective estimation of the delay time distribution in maintenance modeling, *European Journal of Operational Research* **99(3)**, 1997, 516-529.
Wu, F.G., Lee, Y.J., Lin, M.C., Using the fuzzy analytic hierarchy process on optimum spatial allocation, *International Journal of Industrial Ergonomics* **33**, 2004, 553-569.
Ye, S., Tiong, R.L.K., NPV-AT-METHOD in infrastructure project investment evaluation, *Journal of construction engineering and management*, 2000, 227-233.
Yu, C.S., A GP-AHP method for solving group decision-making fuzzy AHP problems, *Computers & Operations Research* **29**, 2002, 1969-2001.
Zadeh L.A., Fuzzy Sets, *Information Control* **8**, 1965, 338-353.
Zahir, S., Geometry of decision making and the vector space formulation of the analytic hierarchy process, *European Journal of Operational Research* **112(2)**, 1999, 373-396.
Zhu, X., Dale, A.P., Java AHP: a web-based decision analysis tool for natural resource and environmental management, *Environmental Modelling & Software* **16**, 2001, 251-262.
Zhu, K.J., Jing, Y., Chang, D.Y., A discussion on extent analysis method and applications of fuzzy AHP, *European Journal of Operational Research* **116(3)**, 1999, 450-456.

Appendix 1

	Proton	Honda	Vauxhall	Volkswagen	Daewoo
Type	Persona	New Civic	Merit	Polo	Lanos
Size of Engine	1600	1600	1600	1600	1600
Central Locking	√	√	√	√	√
Electric Windows	√	√	√	√	√
Power Steeling	√	√	√	√	√
Automatic	√	√	√	√	√
Air Condition	√	√	√	√	√
5 Doors	√	√	√	√	√
Airbags	√	√	√	√	√
Antilock Braking System	×	√	×	√	×
Impact Protection System	×	√	×	√	×
Anti-Theft Devices	×	√	×	√	×
Image	Green	Silver	Metallic blue	Red	Black
Insurance	11	7	8	7	6
Price	£1,850.00	£2,500.00	£2,000.00	£3,000.00	£1,500.00
Car Age	3 years	3 years	3 years	2 years	2 years
Car Mileage	45,000	30,000	25,000	25,000	35,000

Table A-1: Information table

Tables

Intensity of preference (Numerical Value)	Definition (Verbal Scale)	Explanation
1	Equally preferred; Equal preference	Two elements contribute equally to the objective
3	Moderately preferred; Weak preference of one over other	Experience and judgement slightly favour one element over another
5	Strongly preferred; Essential or strong preference	Experience and judgement favour one element over another
7	Very strongly preferred; Demonstrated preference	An element is very strongly favoured and its dominance is demonstrated in practice
9	Extremely preferred; Absolute preference	The evidence favouring one element over another is of the highest possible order of affirmation
2, 4, 6, 8	Intermediate values between the two adjacent judgements	When compromise is needed
Reciprocals of above nonzero	If an element *i* has one of the above numbers assigned to it when compared with element *j*, then *j* has the reciprocal value when compared with *i*.	
Ratios	Ratios arising from the scale	If consistency were to be forced by obtaining *n* numerical values to span the matrix.

Table 1: Scale of relative preference based on Saaty (1980)

	C_1	C_2	C_3	C_4	C_5
C_1	1	3	-	3	1/5
C_2	1/3	1	1/9	-	1/2
C_3	-	9	1	7	1
C_4	1/3	-	1/7	1	1/5
C_5	5	2	1	5	1

Table 2: Pairwise comparisons between criteria based on the DM’s opinions

a) C_1	A_1	A_2	A_3	A_4	A_5
A_1	1	-	-	-	5
A_2	-	1	7	1/3	5
A_3	-	1/7	1	1/5	5
A_4	-	3	5	1	5
A_5	1/5	1/5	1/5	1/5	1

b) C_2	A_1	A_2	A_3	A_4	A_5
A_1	1	1/8	1/8	1/7	-
A_2	8	1	-	-	5
A_3	8	-	1	1/8	5
A_4	7	-	8	1	5
A_5	-	1/5	1/5	1/5	1

c) C_3	A_1	A_2	A_3	A_4	A_5
A_1	1	1/8	-	1/7	-
A_2	8	1	-	1	3
A_3	-	-	1	1/8	3
A_4	7	1	8	1	3
A_5	-	1/3	1/3	1/3	1

d) C_4	A_1	A_2	A_3	A_4	A_5
A_1	1	1/7	1/5	1/9	-
A_2	7	1	7	-	1/6
A_3	5	1/7	1	1/8	6
A_4	9	-	8	1	6
A_5	-	6	1/6	1/6	1

e) C_5	A_1	A_2	A_3	A_4	A_5
A_1	1	3	2	5	1/3
A_2	1/3	1	1/3	3	1/5
A_3	1/2	3	1	4	1/3
A_4	1/5	1/3	1/4	1	1/9
A_5	3	5	3	9	1

Table 3. Pairwise comparisons between alternatives over the different criteria.

	C_1	C_2	C_3	C_4	C_5
C_1	(1, 1, 1)	$(3-\delta, 3, 3+\delta)$	-	$(3-\delta, 3, 3+\delta)$	$(1/5-\delta, 1/5, 1/5+\delta)$
C_2	$(1/3-\delta, 1/3, 1/3+\delta)$	(1, 1, 1)	$(1/9-\delta, 1/9, 1/9+\delta)$	-	$(1/2-\delta, 1/2, 1/2+\delta)$
C_3	-	$(9-\delta, 9, 9+\delta)$	(1, 1, 1)	$(7-\delta, 7, 7+\delta)$	$(\frac{1}{1+\delta}, 1, 1+\delta)$
C_4	$(1/3-\delta, 1/3, 1/3+\delta)$	-	$(1/7-\delta, 1/7, 1/7+\delta)$	(1, 1, 1)	$(1/5-\delta, 1/5, 1/5+\delta)$
C_5	$(5-\delta, 5, 5+\delta)$	$(2-\delta, 2, 2+\delta)$	$(\frac{1}{1+\delta}, 1, 1+\delta)$	$(5-\delta, 5, 5+\delta)$	(1, 1, 1)

Table 4: The fuzzy comparison matrix based on using the 1-9 scale

	C_1	C_2	C_3	C_4	C_5
C_1	(1, 1, 1)	(2.5, 3, 3 .5)	-	(2.5, 3, 3 .5)	(0.1818, 1/5, 0.2222)
C_2	(0.2857, 1/3, 0.4)	(1, 1, 1)	(1/9, 1/9, 0.1176)	-	(0.4, 0.5, 0.6667)
C_3	-	(8.5, 9, 9.5)	(1, 1, 1)	(6.5, 7, 7 .5)	(0.6667, 1, 1.5)
C_4	(0.2857, 1/3, 0.4)	-	(0.1333, 1/7, 0.1538)	(1, 1, 1)	(0.1818, 1/5, 0.2222)
C_5	(4.5, 5, 5.5)	(1.5, 2, 2.5)	(0.6667, 1, 1.5)	(4.5, 5, 5.5)	(1, 1, 1)

Table 5: The fuzzy comparison matrix over different criteria where $\delta = 0.5$

	Row Sums	Column Sums
C_1	(6.1818, 7.2000, 8.2222)	(6.0714, 6.6667, 7.3000)
C_2	(1.7968, 1.9444, 2.1843)	(13.500, 15.000, 16.000)
C_3	(16.6667, 18.000, 19.000)	(1.9111, 2.2540, 2.7715)
C_4	(1.6001, 1.6762, 1.7761)	(14.500, 16.000, 17.500)
C_5	(12.1667, 14.000, 16.000)	(2.4303, 2.9000, 3.6111)
Sum of column sums		(38.4128, 42.8207, 47.1826)

Table 6: Sum of rows and columns based on different criteria

	Weight values for DAs					Weight values for criteria
DAs	A_1	A_2	A_3	A_4	A_5	
C_1	0	0.4641	0	0.5359	0	0
C_2	0	0	0	1	0	0
C_3	0	0	0	1	0	0.7123
C_4	0	0	0	1	0	0
C_5	0	0	0	0	1	0.2877
Final Results	0	0	0	0.7123	0.2877	
Ranking orders	$[A_4, A_5, A_1 = A_2 = A_3]$					

Table 7: The sets of weight values for all fuzzy comparison matrices and the final results obtained where $\delta = 0.5$ based on the DM's opinions

	Weight values for DAs					Weight values for criteria
DAs	A_1	A_2	A_3	A_4	A_5	
C_1	0.1919	0.2412	0.1988	0.2436	0.1244	0.2485
C_2	0.0117	0.2919	0.2930	0.3294	0.0740	0.0625
C_3	0.0023	0.3385	0.1943	0.3795	0.0854	0.3186
C_4	0.0128	0.2590	0.2427	0.2965	0.1890	0.0678
C_5	0.2464	0.1730	0.2237	0.0710	0.2859	0.3026
Final Results	0.1246	0.2560	0.2138	0.2436	0.1621	
Ranking orders	$[A_2, A_4, A_3, A_5, A_1]$					

Table 8: The sets of weight values for all fuzzy comparison matrices and the final results obtained where δ =4.73 based on the DM's opinions

Figures

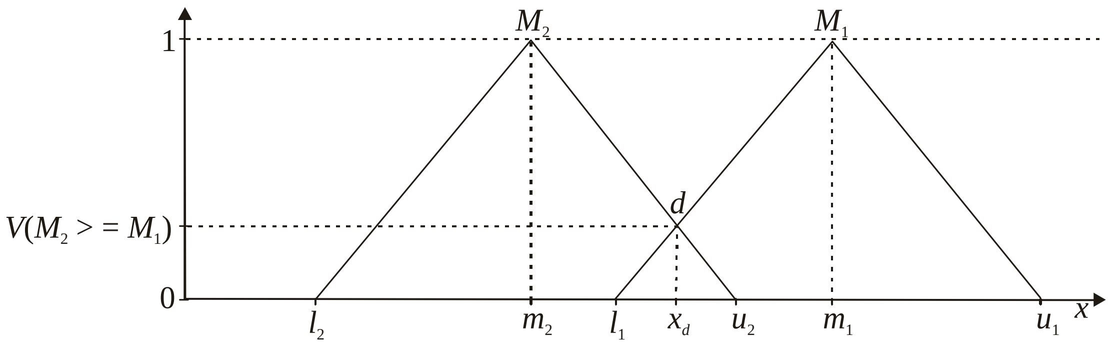

Figure 1: The comparison of two fuzzy numbers M_1 and M_2

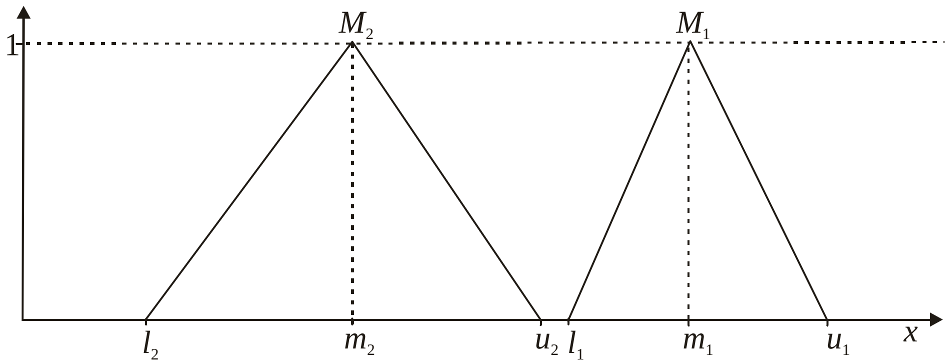

Figure 2: Comparison of two fuzzy numbers M_1 and M_2 while $l_1 > u_2$

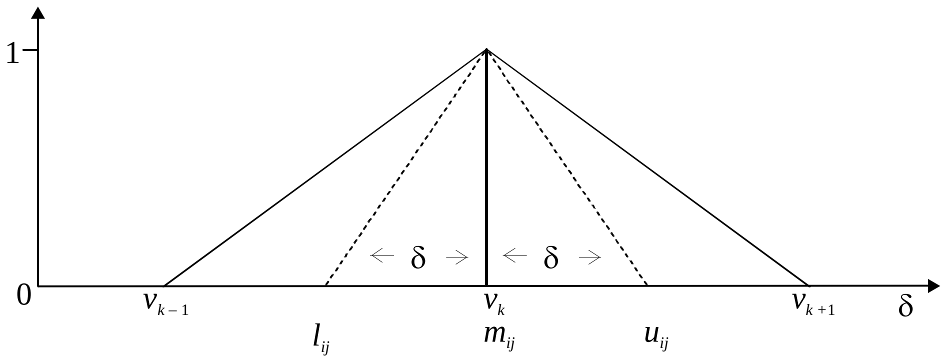

Figure 3: Description of the degree of fuzziness δ according to Zhu et al. (1999)

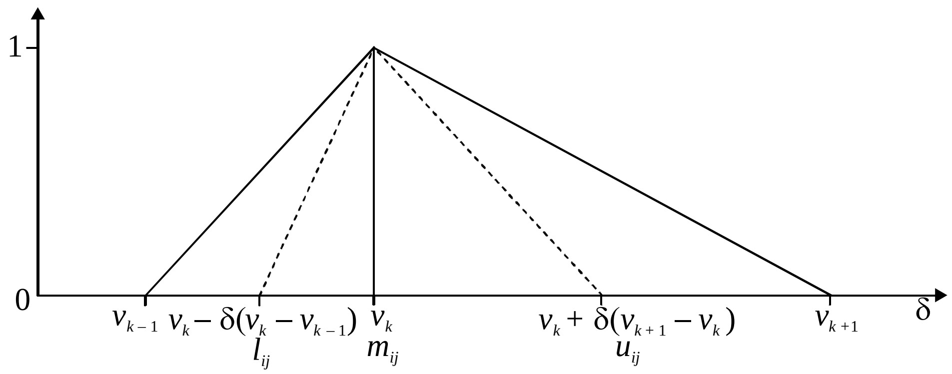

Figure 4: Representation of fuzzy number (l_{ij}, m_{ij}, u_{ij}) based on δ as a proportion between the successive scale values

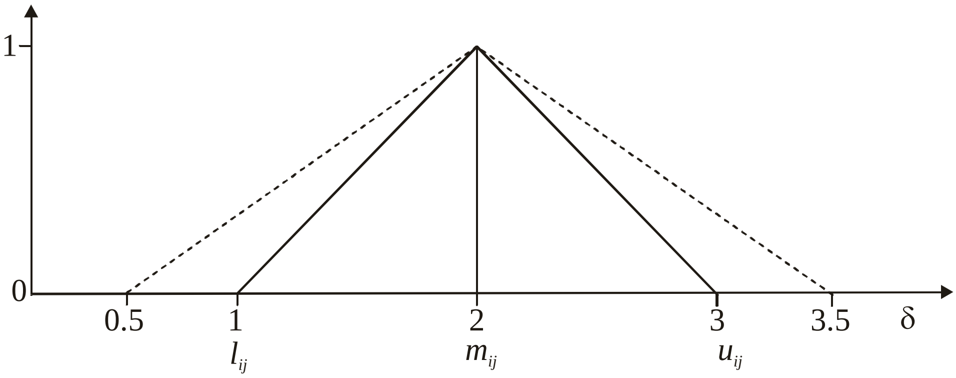

Figure 5: The example for $m_{ij} = v_k = 2$ while $\delta = 1.5$

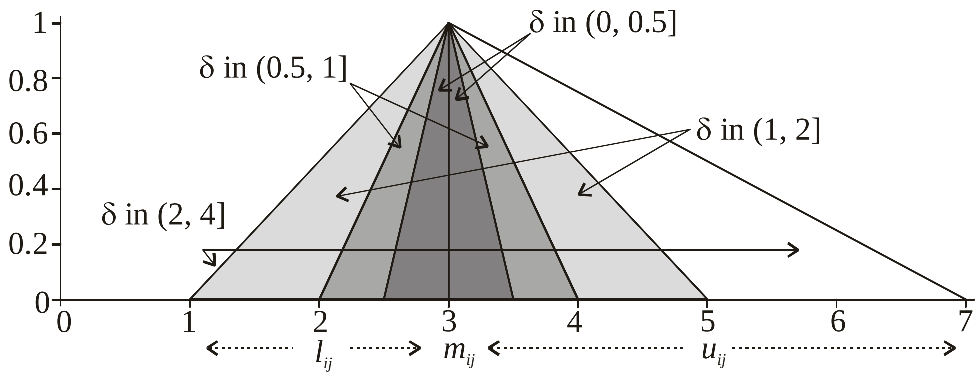

Figure 6: Illustration of l_{ij} and u_{ij} bounds on the fuzzy number depending on changes in the δ value

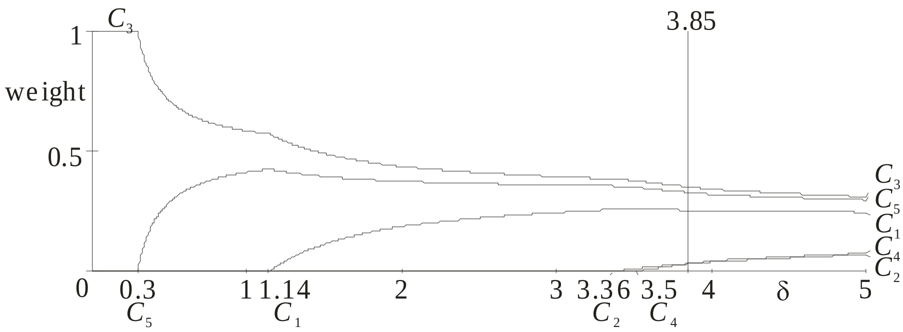

Figure 7: The weights between criteria based on the DM's opinions for $0 \leq \delta \leq 5$

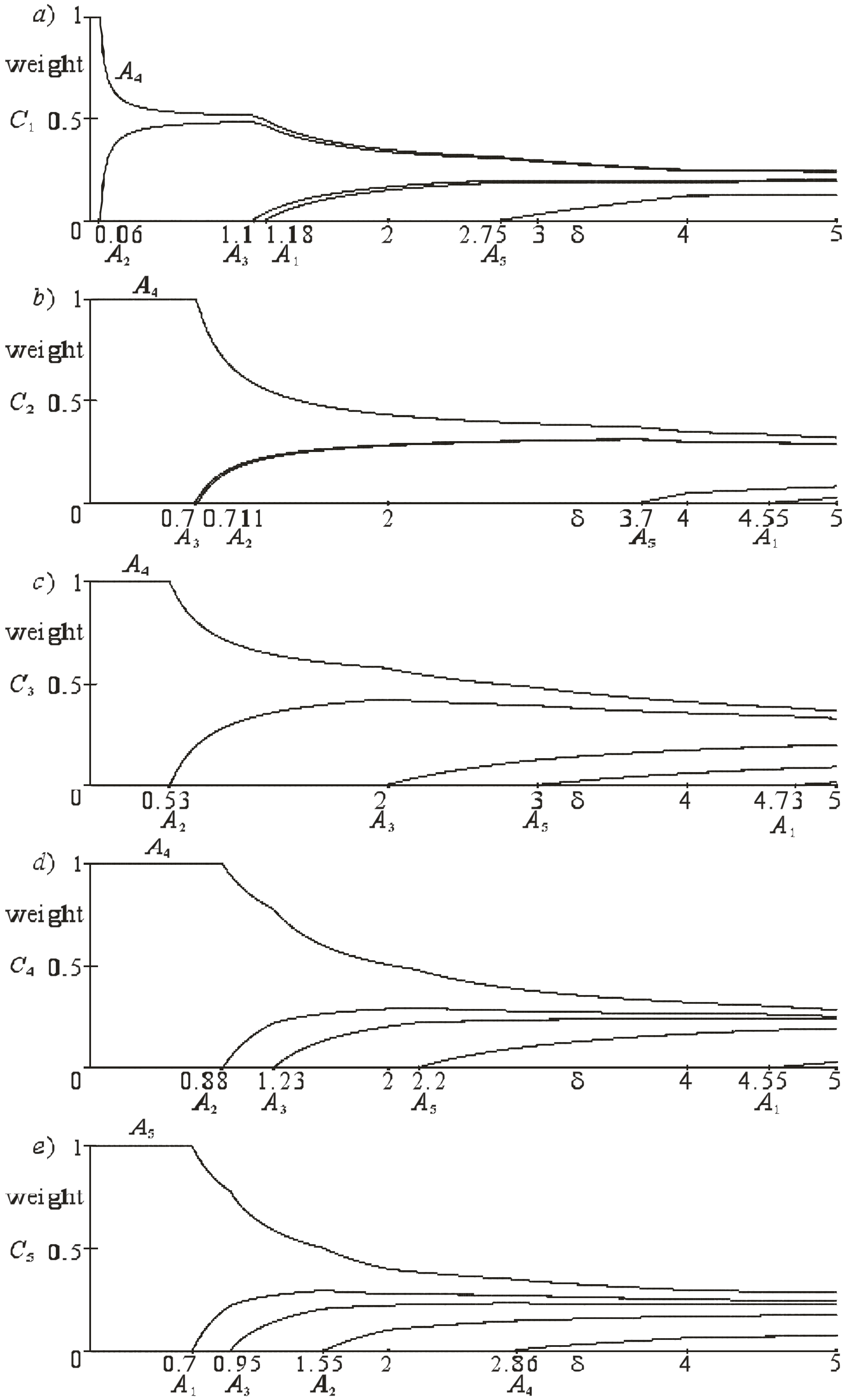

Figure 8: The weights between alternatives based on C_1 to C_5