# A Dynamic Heterogeneous Beliefs CAPM

Carl Chiarella
School of Finance and Economics
University of Technology Sydney, Australia

Roberto Dieci
Dipartimento di Matematica per le Scienze Economiche e Sociali
University of Bologna, Italy

Xue-Zhong He
School of Finance and Economics
University of Technology Sydney, Australia

May 6, 2006

**Abstract**

We reconsider the derivation of the traditional capital asset pricing model (CAPM) in the discrete time setting for a portfolio of one risk-free asset and many risky assets. In contrast to the standard setting we consider heterogeneous agents whose expectations of future returns based on statistical properties of past returns induce expectations feedback. We assume that agents formulate their demands based upon heterogeneous beliefs about conditional means and covariances of the risky asset returns, and that the risky returns evolve over time via the determination of market clearing prices, under a Walrasian auctioneer scenario. In this framework we first construct a 'consensus' belief (with respect to the means and covariances of the risky asset returns) to represent the aggregate market belief and derive a heterogeneous CAPM which relates aggregate excess return on risky assets with aggregate excess return on the market portfolio via an aggregate beta coefficient for risky assets. We then adopt this perspective to establish a 'market fraction' model in which agents are grouped according to their beliefs. The impact of different beliefs on the market equilibrium returns and the beta-coefficients is analysed. In particular, we focus on some "classical" heterogeneous agents types - fundamentalists, trend followers and noise traders - and investigate how the key behavioral parameters affect the time varying behaviour of the aggregate beta coefficient.

## 1 Introduction

...

Section 2 derives equilibrium CAPM-like relationships for asset returns in the case of heterogeneous beliefs and relates a 'consensus' belief about the expected excess return on each risky asset to a 'consensus' belief about expected market return, via aggregate beta coefficients. Section 3 discusses further the wealth dynamics and the beta coefficients, and relates them to the heterogeneous beliefs about the returns on the risky assets. Section 4 considers explicitly the supply of the risky securities, introduces market clearing conditions to derive equilibrium prices, and relates the aggregate beta coefficients to the market clearing prices. Section 5 combines this setup with a dynamic, "market fraction" multi-asset framework with heterogeneous groups of agents, which generalizes earlier contributions by Brock and Hommes (1998) and Chiarella and He (2001a,b), and highlights how the aggregate beta coefficients may vary over time once agents' beliefs are assumed to be updated dynamically at each time step as a function of past realized returns. This framework is then specialized to the case of interaction between fundamentalists, trend followers and noise traders in Section 6, which reports the results of numerical simulations and investigates how the dynamics of market returns and aggregate beta coefficients is affected by the key behavioural parameters. Section 7 concludes.

## 2 Derivation of Heterogeneous Beliefs CAPM in terms of asset returns

The present section generalizes the derivation of CAPM relationships - as carried out for instance by Huang and Litzenberger (1988) Section 4.15 - to the case of investors with heterogeneous beliefs about asset returns. Some of the ideas contained in the present section are adapted from Lintner (1969), where aggregation of individual assessments about future payoffs is performed in a mean-variance framework.

Assume that there exists a riskless asset with a rate of return $r_f$ and that the rates of return $\widetilde{r}_j$, $j = 1, 2, ..., N$, of the risky assets are multivariate normally distributed. Assume that investor $i$, $i = 1, 2, ..., I$, has a concave and strictly increasing utility of wealth function $u_i(\cdot)$. Following Huang-Litzenberger (Section 4.15) the optimally invested random wealth of investor $i$ (at the end of the period) satisfies

$$E_i\left[u_i'(\widetilde{W}_i)\right] E_i\left[\widetilde{r}_j - r_f\right] = -E_i\left[u_i''(\widetilde{W}_i)\right] Cov_i(\widetilde{W}_i, \widetilde{r}_j) \tag{1}$$

for any $j = 1, 2, ..., N$, where

$$\widetilde{W}_i = W_0^i \left(1 + r_f + \sum_{j=1}^{N} w_{ij}(\widetilde{r}_j - r_f)\right)$$

and $w_{ij}$ is the fraction of wealth of agent $i$ invested in the risky asset $j$. By defining the $i$-th investor's *global absolute risk aversion*

$$\theta_i := \frac{-E_i\left[u_i''(\widetilde{W}_i)\right]}{E_i\left[u_i'(\widetilde{W}_i)\right]}$$

condition (1) becomes

$$\theta_i^{-1} E_i\left[\widetilde{r}_j - r_f\right] = Cov_i(\widetilde{W}_i, \widetilde{r}_j) \qquad j = 1, 2, ..., N \tag{2}$$

Note that

$$Cov_i(\widetilde{W}_i, \widetilde{r}_j) = Cov_i\left[W_0^i\left(\sum_{k=1}^{N} w_{ik}\widetilde{r}_k\right), \widetilde{r}_j\right] = W_0^i \sum_{k=1}^{N} w_{ik} Cov_i\left(\widetilde{r}_k, \widetilde{r}_j\right)$$

It follows that the conditions (2) can be rewritten with vector notation as

$$\theta_i^{-1} E_i\left[\widetilde{\mathbf{r}} - r_f \mathbf{1}\right] = W_0^i \mathbf{\Omega}_i \mathbf{w}_i \tag{3}$$

where $\widetilde{\mathbf{r}} = [\widetilde{r}_1, \widetilde{r}_2, ..., \widetilde{r}_N]^\top$, $\mathbf{1} = [1, 1, ..., 1]^\top$, $\mathbf{w}_i = [w_{i1}, w_{i2}, ..., w_{iN}]^\top$, and $\mathbf{\Omega}_i = [\sigma_{i,jk}]$, $j, k = 1, 2, ..., N$, where $\sigma_{i,jk} := Cov_i\left(\widetilde{r}_j, \widetilde{r}_k\right)$. Equation (3) can be rewritten as

$$W_0^i \mathbf{w}_i = \theta_i^{-1} \mathbf{\Omega}_i^{-1} E_i\left[\widetilde{\mathbf{r}} - r_f \mathbf{1}\right] \tag{4}$$

and summing across $i$

$$\sum_{i=1}^{I} W_0^i \mathbf{w}_i = \sum_{i=1}^{I} \theta_i^{-1} \mathbf{\Omega}_i^{-1} E_i\left[\widetilde{\mathbf{r}} - r_f \mathbf{1}\right]$$

We remark that the optimal investment policy of agent $i$ is only implicitly defined by (4), because in general $\theta_i = \theta_i(\mathbf{w}_i)$ will depend, in its turn, on $\mathbf{w}_i$. Nevertheless, at this stage we are interested in equilibrium relationships involving aggregate beliefs, which do not require $\mathbf{w}_i$ to be made explicit. By defining the vector of the *aggregate wealth proportions* invested in the risky assets

$$\mathbf{w}_a := \frac{1}{W_{m0}} \sum_{i=1}^{I} W_0^i \mathbf{w}_i \tag{5}$$

where $W_{m0} := \sum_{i=1}^{I} W_0^i$ represents total initial wealth, one obtains

$$\mathbf{w}_a = \frac{1}{W_{m0}} \sum_{i=1}^{I} \theta_i^{-1} \mathbf{\Omega}_i^{-1} E_i\left[\widetilde{\mathbf{r}} - r_f \mathbf{1}\right] \tag{6}$$

An "aggregate" variance/covariance matrix $\mathbf{\Omega}_a$, which represents a 'consensus' belief about the covariance structure of returns, can be defined as a weighted harmonic mean of the individual covariance matrices $\mathbf{\Omega}_i$, such that

$$\mathbf{\Omega}_a^{-1} = \Theta \sum_{i=1}^{I} \theta_i^{-1} \mathbf{\Omega}_i^{-1} \tag{7}$$

where $\Theta := \left(\sum_{i=1}^{I}\theta_i^{-1}\right)^{-1}$. Then it follows from (6) that

$$\begin{aligned}\mathbf{w}_a &= \frac{1}{W_{m0}}\left[\sum_{i=1}^{I}\theta_i^{-1}\boldsymbol{\Omega}_i^{-1}E_i\left(\widetilde{\mathbf{r}}\right) - \Theta^{-1}\boldsymbol{\Omega}_a^{-1}r_f\mathbf{1}\right] = \\ &= \frac{1}{\Theta W_{m0}}\boldsymbol{\Omega}_a^{-1}\left[\Theta\boldsymbol{\Omega}_a\sum_{i=1}^{I}\theta_i^{-1}\boldsymbol{\Omega}_i^{-1}E_i\left(\widetilde{\mathbf{r}}\right) - r_f\mathbf{1}\right]\end{aligned}$$

from which

$$\boldsymbol{\Omega}_a\mathbf{w}_a = \frac{1}{\Theta W_{m0}}\left[\Theta\boldsymbol{\Omega}_a\sum_{i=1}^{I}\theta_i^{-1}\boldsymbol{\Omega}_i^{-1}E_i\left(\widetilde{\mathbf{r}}\right) - r_f\mathbf{1}\right] \tag{8}$$

Note that $\Theta W_{m0}$ can be interpreted as the *aggregate relative risk aversion* of the economy in equilibrium. Similarly to Huang and Litzenberger (1988), Section 4.15, we can define the random market return $\widetilde{r}_m$ as the one which satisfies

$$\widetilde{W}_m := \sum_{i=1}^{I}\widetilde{W}_i = W_{m0}(1+\widetilde{r}_m) \quad \text{i.e.} \quad \widetilde{r}_m = \frac{\widetilde{W}_m}{W_{m0}} - 1$$

where $\widetilde{W}_m$ represents the random end-of-period wealth in the market. Note that $\widetilde{r}_m$ can also be rewritten in terms of aggregate wealth proportions as

$$\widetilde{r}_m := r_f + \mathbf{w}_a^{\top}\left(\widetilde{\mathbf{r}} - r_f\mathbf{1}\right)$$

The quantity $\sigma_{a,m}^2 := \mathbf{w}_a^{\top}\boldsymbol{\Omega}_a\mathbf{w}_a$ can be interpreted as the aggregate 'consensus' belief about the variance of the market return, given by

$$\sigma_{a,m}^2 := \mathbf{w}_a^{\top}\boldsymbol{\Omega}_a\mathbf{w}_a = \frac{1}{\Theta W_{m0}}\left[\mathbf{w}_a^{\top}\Theta\boldsymbol{\Omega}_a\sum_{i=1}^{I}\theta_i^{-1}\boldsymbol{\Omega}_i^{-1}E_i\left(\widetilde{\mathbf{r}}\right) - \mathbf{w}_a^{\top}r_f\mathbf{1}\right] \tag{9}$$

Next, define the "aggregate" expected returns on the risky assets $E_a\left(\widetilde{\mathbf{r}}\right)$ (consensus beliefs about expected returns) as a weighted average of the individual expected returns $E_i\left(\widetilde{\mathbf{r}}\right)$, such that

$$\sum_{i=1}^{I}\theta_i^{-1}\boldsymbol{\Omega}_i^{-1}E_i\left(\widetilde{\mathbf{r}}\right) = \sum_{i=1}^{I}\theta_i^{-1}\boldsymbol{\Omega}_i^{-1}E_a\left(\widetilde{\mathbf{r}}\right)$$

from which

$$E_a\left(\widetilde{\mathbf{r}}\right) = \Theta\boldsymbol{\Omega}_a\sum_{i=1}^{I}\theta_i^{-1}\boldsymbol{\Omega}_i^{-1}E_i\left(\widetilde{\mathbf{r}}\right) \tag{10}$$

Define also the aggregate expected market return

$$E_a(\widetilde{r}_m) := r_f + \mathbf{w}_a^{\top}\left(E_a\left(\widetilde{\mathbf{r}}\right) - r_f\mathbf{1}\right) \tag{11}$$

Equation (9) becomes

$$\sigma_{a,m}^2 = \frac{1}{\Theta W_{m0}} \left\{ \mathbf{w}_a^\top \left[ E_a \left( \widetilde{\mathbf{r}} \right) - r_f \mathbf{1} \right] \right\} \tag{12}$$

which implies that

$$[E_a(\widetilde{r}_m) - r_f] = \Theta W_{m0} \sigma_{a,m}^2 \tag{13}$$

i.e. the aggregate expected market risk premium is proportional to the aggregate relative risk aversion of the economy.

It follows from (8), (10) and (13) that

$$\frac{1}{\sigma_{a,m}^2} \mathbf{\Omega}_a \mathbf{w}_a = \frac{1}{E_a(\widetilde{r}_m) - r_f} \left[ E_a \left( \widetilde{\mathbf{r}} \right) - r_f \mathbf{1} \right]$$

i.e.

$$\left[ E_a \left( \widetilde{\mathbf{r}} \right) - r_f \mathbf{1} \right] = \frac{E_a(\widetilde{r}_m) - r_f}{\sigma_{a,m}^2} \mathbf{\Omega}_a \mathbf{w}_a \tag{14}$$

The entries of $\mathbf{\Omega}_a \mathbf{w}_a$ represent the aggregate covariances between the return on each risky asset and the market return, i.e.

$$\mathbf{\Omega}_a \mathbf{w}_a = [\sigma_{a,jm}] \qquad j = 1, 2, ..., N$$

where $\sigma_{a,jm} := Cov_a(\widetilde{r}_j, \widetilde{r}_m)$, so that (14) can be rewritten component by component as

$$E_a(\widetilde{r}_j) - r_f = \frac{\sigma_{a,jm}}{\sigma_{a,m}^2} [E_a(\widetilde{r}_m) - r_f] \qquad j = 1, 2, ..., N \tag{15}$$

where $\sigma_{a,jm}/\sigma_{a,jm} := \beta_{a,jm}$ represents the aggregate beta coefficient of the $j$-th risky asset. Equation (15) is the CAPM relation, in terms of returns, generalized to the case of heterogeneous beliefs. The vector $\boldsymbol{\beta}_{a,m} := [\beta_{a,1m}, \beta_{a,2m}, ..., \beta_{a,Nm}]^\top$ of the aggregate beta coefficients in (14) is thus given by

$$\boldsymbol{\beta}_{a,m} = \frac{1}{\sigma_{a,m}^2} \mathbf{\Omega}_a \mathbf{w}_a \tag{16}$$

# 3 Heterogeneous beliefs, wealth dynamics and beta coefficients

It is convenient to rewrite the beta coefficients (16) in a different form, which stresses the way they depend on agents' aggregate beliefs about random returns and initial wealth. Define

$$\boldsymbol{\zeta}_i := \mathbf{w}_i W_0^i = \theta_i^{-1} \mathbf{\Omega}_i^{-1} E_i \left[ \widetilde{\mathbf{r}} - r_f \mathbf{1} \right]$$

as the vector of optimal dollar investment in each risky asset by agent $i$, and

$$\boldsymbol{\zeta} := \sum_{i=1}^{I} \mathbf{w}_i W_0^i = \sum_{i=1}^{I} \theta_i^{-1} \mathbf{\Omega}_i^{-1} E_i \left[ \widetilde{\mathbf{r}} - r_f \mathbf{1} \right] \tag{17}$$

the vector which collects the aggregate dollar demands for each risky asset. Using (10), $\boldsymbol{\zeta}$ can be rewritten as a function of the aggregate beliefs, as follows:

$$\boldsymbol{\zeta} = \Theta^{-1}\boldsymbol{\Omega}_a^{-1}E_a\left[\widetilde{\mathbf{r}} - r_f\mathbf{1}\right] \tag{18}$$

Note also that:

$$\boldsymbol{\zeta} = \mathbf{w}_a W_{m0} \tag{19}$$

where $\mathbf{w}_a$ is the vector of aggregate wealth proportions invested in the risky assets, defined by (5).

We now focus on the wealth dynamics. Denote by $\widetilde{W}_m := \sum_{i=1}^{I}\widetilde{W}_i$ the random end-of-period wealth in the market. Using (18), the optimally invested final wealth is given by

$$\begin{aligned}\widetilde{W}_m &= W_{m0}(1+r_f) + \sum_{i=1}^{I} W_0^i \mathbf{w}_i^\top(\widetilde{\mathbf{r}} - r_f\mathbf{1}) = W_{m0}(1+r_f) + \boldsymbol{\zeta}^\top(\widetilde{\mathbf{r}} - r_f\mathbf{1}) = \\ &= W_{m0}(1+r_f) + \Theta^{-1}[E_a(\widetilde{\mathbf{r}}) - r_f\mathbf{1}]^\top\boldsymbol{\Omega}_a^{-1}(\widetilde{\mathbf{r}} - r_f\mathbf{1})\end{aligned}$$

and the aggregate expectation and variance/covariance beliefs about the excess market payoff $\widetilde{W}_m - W_{m0}(1+r_f)$ are

$$\begin{aligned}E_a[\widetilde{W}_m - W_{m0}(1+r_f)] &= \boldsymbol{\zeta}^\top[E_a(\widetilde{\mathbf{r}}) - r_f\mathbf{1}] \\ &= \Theta^{-1}[E_a(\widetilde{\mathbf{r}}) - r_f\mathbf{1}]^\top\boldsymbol{\Omega}_a^{-1}[E_a(\widetilde{\mathbf{r}}) - r_f\mathbf{1}]\end{aligned} \tag{20}$$

$$\begin{aligned}Var_a[\widetilde{W}_m - W_{m0}(1+r_f)] &= Var_a[\widetilde{W}_m] = \boldsymbol{\zeta}^\top\boldsymbol{\Omega}_a\boldsymbol{\zeta} \\ &= \Theta^{-2}[E_a(\widetilde{\mathbf{r}}) - r_f\mathbf{1}]^\top\boldsymbol{\Omega}_a^{-1}[E_a(\widetilde{\mathbf{r}}) - r_f\mathbf{1}]\end{aligned} \tag{21}$$

from which one gets

$$\Theta = \frac{E_a[\widetilde{W}_m - W_{m0}(1+r_f)]}{Var_a[\widetilde{W}_m - W_{m0}(1+r_f)]} = \frac{1}{W_{m0}}\frac{E_a(\widetilde{r}_m) - r_f}{\sigma_{a,m}^2}$$

which is obviously equivalent to (13), since

$$E_a(\widetilde{r}_m) - r_f = \frac{1}{W_{m0}}E_a[\widetilde{W}_m - W_{m0}(1+r_f)]$$

$$\sigma_{a,m}^2 := Var_a(\widetilde{r}_m) = \frac{1}{W_{m0}^2}Var_a[\widetilde{W}_m - W_{m0}(1+r_f)]$$

Using (20) and (21) the latter equations can be rewritten, respectively, as

$$E_a(\widetilde{r}_m) - r_f = \frac{1}{\Theta W_{m0}}[E_a(\widetilde{\mathbf{r}}) - r_f\mathbf{1}]^\top\boldsymbol{\Omega}_a^{-1}[E_a(\widetilde{\mathbf{r}}) - r_f\mathbf{1}] \tag{22}$$

$$\sigma_{a,m}^2 := Var_a(\widetilde{r}_m) = \frac{1}{(\Theta W_{m0})^2}[E_a(\widetilde{\mathbf{r}}) - r_f\mathbf{1}]^\top\boldsymbol{\Omega}_a^{-1}[E_a(\widetilde{\mathbf{r}}) - r_f\mathbf{1}] \tag{23}$$

Therefore, using (19), (18), and (23), the beta coefficients (16) can be rewritten as

$$\boldsymbol{\beta}_{a,m} = \frac{\Theta W_{m0}}{[E_a(\widetilde{\mathbf{r}}) - r_f\mathbf{1}]^\top \boldsymbol{\Omega}_a^{-1}[E_a(\widetilde{\mathbf{r}}) - r_f\mathbf{1}]}[E_a(\widetilde{\mathbf{r}}) - r_f\mathbf{1}] \tag{24}$$

An important remark about Equation (24) is that it provides the beta coefficients as functions of the beliefs about returns. However, equation (24) also depends on the initial wealth level. Note that we have not made any assumption up to now about the net supply of assets in the market, and therefore about the composition of the total market wealth. The usual assumption in the CAPM literature is that the riskless asset is in zero net supply in the market, which can be formalized as

$$\sum_{i=1}^{I} W_0^i \left(1 - \sum_{j=1}^{N} w_{ij}\right) = 0$$

i.e.

$$W_{m0} := \sum_{i=1}^{I} W_0^i = \sum_{i=1}^{I}\sum_{j=1}^{N} W_0^i w_{ij} = \left(\sum_{i=1}^{I} W_0^i \mathbf{w}_i^\top\right)\mathbf{1} = \boldsymbol{\zeta}^\top\mathbf{1}$$

where $\boldsymbol{\zeta}$ is the dollar demand vector for the risky assets, defined by (17). Under these assumptions, the total current market wealth is exactly equal to the total wealth invested in the risky assets[1], i.e.

$$W_{m0} = \boldsymbol{\zeta}^\top\mathbf{1} = \Theta^{-1}[E_a(\widetilde{\mathbf{r}}) - r_f\mathbf{1}]^\top \boldsymbol{\Omega}_a^{-1}\mathbf{1} \tag{25}$$

whereas the final wealth $\widetilde{W}_m$ is given by

$$\widetilde{W}_m = \boldsymbol{\zeta}^\top(\mathbf{1}+\widetilde{\mathbf{r}}) = \Theta^{-1}[E_a(\widetilde{\mathbf{r}}) - r_f\mathbf{1}]^\top \boldsymbol{\Omega}_a^{-1}(\mathbf{1}+\widetilde{\mathbf{r}}) \tag{26}$$

As a consequence, the rate of return on total market wealth $\widetilde{r}_m := \widetilde{W}_m/W_{m0} - 1$ can be rewritten as

$$\widetilde{r}_m = \frac{[E_a(\widetilde{\mathbf{r}}) - r_f\mathbf{1}]^\top \boldsymbol{\Omega}_a^{-1}\widetilde{\mathbf{r}}}{[E_a(\widetilde{\mathbf{r}}) - r_f\mathbf{1}]^\top \boldsymbol{\Omega}_a^{-1}\mathbf{1}} \tag{27}$$

and represents the return on the "market portfolio" of the risky assets, which has weights summing up to unity, given by

$$\mathbf{w}_a := \frac{1}{W_{m0}}\sum_{i=1}^{I} W_0^i \mathbf{w}_i, \quad \mathbf{w}_a^\top\mathbf{1} = 1$$

Using (25), the beta coefficients with respect to the "market portfolio of the risky assets" can thus be rewritten as a function of the beliefs about the returns of the risky assets, as follows

$$\boldsymbol{\beta}_{a,m} = \frac{[E_a(\widetilde{\mathbf{r}}) - r_f\mathbf{1}]^\top \boldsymbol{\Omega}_a^{-1}\mathbf{1}}{[E_a(\widetilde{\mathbf{r}}) - r_f\mathbf{1}]^\top \boldsymbol{\Omega}_a^{-1}[E_a(\widetilde{\mathbf{r}}) - r_f\mathbf{1}]}[E_a(\widetilde{\mathbf{r}}) - r_f\mathbf{1}] \tag{28}$$

[1] Note that this is not restrictive under CARA utility - which will be assumed throughout the next sections - because in this case the amount of wealth invested in the risky assets is independent on the wealth level, and thus in a dynamic setting the total amount of wealth invested in the riskless asset has no effect on the price and the return dynamics of the risky securities.

# 4 Equilibrium prices

In the following we assume that agents have CARA utility of wealth functions, so that the global absolute risk aversion of agent $i$, $\theta_i = -E_i\left[u_i''(\widetilde{W}_i)\right]/E_i\left[u_i'(\widetilde{W}_i)\right]$, and the aggregate risk aversion $\Theta$, are constant. Note that in this case (3) allows to obtain explicitly the optimal demand of each agent

$$\mathbf{w}_i = \frac{1}{W_0^i}\theta_i^{-1}\mathbf{\Omega}_i^{-1}E_i\left[\widetilde{\mathbf{r}} - r_f\mathbf{1}\right] \tag{29}$$

Denote now by $\mathbf{p}_0 = [p_{01}, p_{02}, ..., p_{0N}]^\top$ the vector of current prices, to be determined by the market clearing conditions under a Walrasian auctioneer scenario, and by $\mathbf{z} :=[z_1, z_2, ..., z_N]^T$ the (positive) supply vector (number of shares). Let $\mathbf{Z} := diag[z_1, z_2, ..., z_N]$ a $(N \times N)$ diagonal matrix whose entries are the elements of $\mathbf{z}$. Simultaneous market clearing for all the risky assets implies

$$\boldsymbol{\zeta} = \mathbf{Z}\mathbf{p}_0 \tag{30}$$

i.e.

$$\mathbf{p}_0 = \mathbf{Z}^{-1}\Theta^{-1}\mathbf{\Omega}_a^{-1}\left[E_a\left(\widetilde{\mathbf{r}}\right) - r_f\mathbf{1}\right] \tag{31}$$

Note that we have extracted market clearing prices by taking the view that during the process of market equilibration, the investors' beliefs about the joint distribution of rates of returns are fixed, i.e. independent of current market prices. Of course Equation (31) can be expressed also in terms of the individual beliefs and attitudes of the $I$ agents:

$$\mathbf{p}_0 = \mathbf{Z}^{-1}\sum_{i=1}^{I}\theta_i^{-1}\mathbf{\Omega}_i^{-1}E_i\left[\widetilde{\mathbf{r}} - r_f\mathbf{1}\right] \tag{32}$$

Note that by using (31) the beta coefficients can also be expressed in terms of market clearing prices, as follows

$$\boldsymbol{\beta}_{a,m} = \frac{\mathbf{p}_0^\top\mathbf{z}}{\mathbf{p}_0^\top\mathbf{Z}\mathbf{\Omega}_a\mathbf{Z}\mathbf{p}_0}\mathbf{\Omega}_a\mathbf{Z}\mathbf{p}_0$$

# 5 A dynamic 'market fraction' multi-asset model

The present section sets up a dynamic heterogeneous beliefs model which extends to a multi-asset framework earlier contributions developed by by Brock and Hommes (1998), Chiarella and He (2001a,b) in the simple case of a single risky security, and thus incorporates into a dynamic setup the CAPM-like return relationships discussed in the previous section in a static framework. To this extent we first rewrite our aggregation relationships in terms of market fractions of heterogeneous agents' types, by introducing some slight changes of notation. Assume that the $I$ investors can be grouped into a finite number of agent-types, indexed by $h \in H$, where the agents within the same group are homogeneous in terms of beliefs $E_h(\widetilde{\mathbf{r}})$ and $\mathbf{\Omega}_h$, as well as risk attitutes $\theta_h$. Denoting by $I_h$, $h \in H$, the number of investors in group $h$, we denote by $n_h := I_h/I$ the fraction of agents of type $h$. We then denote by $\mathbf{s} := (1/I)\mathbf{z}$ the supply of shares per investor. Note that, instead of using the aggregate risk aversion coefficient $\Theta := \left(\sum_{i=1}^{I} \theta_i^{-1}\right)^{-1}$ it is convenient to define the "average" risk aversion $\theta_a$ as follows

$$\theta_a := \left(\sum_{h\in H} n_h \theta_h^{-1}\right)^{-1}$$

where obviously $\theta_a = I\Theta$. Note also that the aggregate beliefs about expected payoffs and variances/covariances can be rewritten, respectively, as follows

$$\mathbf{\Omega}_a = \theta_a^{-1}\left(\sum_{h\in H} n_h \theta_h^{-1}\mathbf{\Omega}_h^{-1}\right)^{-1}$$

$$E_a(\widetilde{\mathbf{r}}) = \theta_a \mathbf{\Omega}_a \sum_{h\in H} n_h \theta_h^{-1}\mathbf{\Omega}_h^{-1} E_h(\widetilde{\mathbf{r}})$$

Finally the equilibrium prices are rewritten as

$$\mathbf{p}_0 = \mathbf{S}^{-1}\theta_a^{-1}\mathbf{\Omega}_a^{-1}\left[E_a\left(\widetilde{\mathbf{r}}\right) - r_f \mathbf{1}\right]$$

or in terms of the beliefs of each group

$$\mathbf{p}_0 = \mathbf{S}^{-1}\sum_{h\in H} n_h \theta_h^{-1}\mathbf{\Omega}_h^{-1}\left[E_h\left(\widetilde{\mathbf{r}}\right) - r_f \mathbf{1}\right]$$

We now turn to the process of formation of heterogeneous beliefs and equilibrium prices in a dynamic setting, from time $t$ to time $t+1$. In doing this, we take the view that agents' beliefs about the returns in the time interval $(t, t+1)$, $\widetilde{\mathbf{r}}_{t+1}$, which are formed before dividends at time $t$ are realized and prices at time $t$ are revealed by the market, determine the aggregate demand for each risky asset at time $t$, which in turns produces the equilibrium prices at time $t$, $\mathbf{p}_t$, via the market clearing conditions. Of course, once prices and dividends at time $t$ are realized, then also the returns $\mathbf{r}_t$ become known. More precisely, we assume that

1) investors' assessments of the end-of-period joint distribution of the returns $\widetilde{\mathbf{r}}_{t+1}$ are formed at time $t$ before the equilibrium prices at time $t$ are determined, and

2) these beliefs remain fixed while the market finds its equilibrium vector of current prices, $\mathbf{p}_t$.

In particular, assumption 1) implies that heterogeneous agents' assessments about $\widetilde{\mathbf{r}}_{t+1}$ are functions of the information up to time $t-1$. We assume in particular that these beliefs can be expressed as functions of the realized returns $\mathbf{r}_{t-1}$, $\mathbf{r}_{t-2}$, ..., i.e., for any group, or belief-type $h \in H$

$$\mathbf{\Omega}_{h,t} := [Cov_{h,t}(\widetilde{r}_{j,t+1}, \widetilde{r}_{k,t+1})] = \mathbf{\Omega}_h(\mathbf{r}_{t-1}, \mathbf{r}_{t-2}, ...) \tag{33}$$

$$E_{h,t}(\widetilde{\mathbf{r}}_{t+1}) = \mathbf{f}_h(\mathbf{r}_{t-1}, \mathbf{r}_{t-2}, ...) \tag{34}$$

where obviously similar representations hold also for the aggregate beliefs $\mathbf{\Omega}_{a,t}$ $:= [Cov_{a,t}(\widetilde{r}_{j,t+1}, \widetilde{r}_{k,t+1})]$ and $E_{a,t}(\widetilde{\mathbf{r}}_{t+1})$.

The market clearing prices at time $t$ become

$$\mathbf{p}_t = \mathbf{S}^{-1} \sum_{h \in H} n_h \theta_h^{-1} \mathbf{\Omega}_{h,t}^{-1} [E_{h,t}(\widetilde{\mathbf{r}}_{t+1}) - r_f \mathbf{1}] \tag{35}$$

(where $\mathbf{\Omega}_{h,t}$ and $E_{h,t}(\widetilde{\mathbf{r}}_{t+1})$ are defined by (33) and (34), respectively), or in terms of the consensus beliefs:

$$\mathbf{p}_t = \mathbf{S}^{-1} \theta_a^{-1} \mathbf{\Omega}_{a,t}^{-1} [E_{a,t}(\widetilde{\mathbf{r}}_{t+1}) - r_f \mathbf{1}] \tag{36}$$

Next, note that the return $r_{j,t}$ on asset $j$, realized over the time interval $(t-1, t)$ is given by

$$r_{j,t} = \frac{p_{j,t} + d_{j,t}}{p_{j,t-1}} - 1$$

where $d_{j,t}$ denotes the realized dividend per share of asset $j$, $j = 1, 2, ..., N$. One can rewrite realized returns with vector notation, as follows

$$\mathbf{r}_t = \mathbf{P}_{t-1}^{-1}(\mathbf{p}_t + \mathbf{d}_t) - \mathbf{1} \tag{37}$$

where $\mathbf{d}_t := [d_{1,t}, d_{2,t}, ..., d_{N,t}]^\top$, and $\mathbf{P}_t := diag(p_{1,t}, p_{2,t}, ..., p_{N,t})$. Equation (37), via the market clearing conditions (35) and the beliefs updating equations (33) and (34), gives the return $\mathbf{r}_t$ as a function of $\mathbf{r}_{t-1}$, $\mathbf{r}_{t-2}$, ... and of the realized dividends $\mathbf{d}_t$ (which will be assumed to follow an exogenous noise process) and thus determine dynamically evolving prices and returns. In the following we will assume an i.i.d. process $\left\{\widetilde{\mathbf{d}}_t\right\}$ for the dividends, with $\overline{\mathbf{d}} := E(\widetilde{\mathbf{d}}_t)$.

We summarize below the dynamical system which describes the market fraction multi-asset model in terms of returns, where the market clearing prices are used as auxiliary variables:

$$\mathbf{r}_t = \mathbf{F}(\mathbf{r}_{t-1}, \mathbf{r}_{t-2}, ...; \widetilde{\mathbf{d}}_t) = \mathbf{P}_{t-1}^{-1}(\mathbf{p}_t + \widetilde{\mathbf{d}}_t) - \mathbf{1}$$

where

$$\mathbf{p}_t = \mathbf{S}^{-1} \sum_{h \in H} n_h \theta_h^{-1} \mathbf{\Omega}_{h,t}^{-1} [E_{h,t}(\widetilde{\mathbf{r}}_{t+1}) - r_f \mathbf{1}]$$

$$\mathbf{P}_t := diag(p_{1,t}, p_{2,t}, ..., p_{N,t})$$

$$\mathbf{\Omega}_{h,t} = \mathbf{\Omega}_h(\mathbf{r}_{t-1}, \mathbf{r}_{t-2}, ...)$$

$$E_{h,t}(\widetilde{\mathbf{r}}_{t+1}) = \mathbf{f}_h(\mathbf{r}_{t-1}, \mathbf{r}_{t-2}, ...)$$

Moreover, at the beginning of each time interval $(t, t+1)$ the aggregate beliefs about returns (based on information up to time $t-1$) satisfy a CAPM-like equation of the type

$$E_{a,t}(\widetilde{\mathbf{r}}_{t+1}) - r_f \mathbf{1} = \boldsymbol{\beta}_{a,mt} [E_{a,t}(\widetilde{r}_{m,t+1}) - r_f]$$

where

$$\widetilde{r}_{m,t+1} = \frac{[E_{a,t}(\widetilde{\mathbf{r}}_{t+1}) - r_f\mathbf{1}]^\top \mathbf{\Omega}_{a,t}^{-1}\widetilde{\mathbf{r}}_{t+1}}{[E_{a,t}(\widetilde{\mathbf{r}}_{t+1}) - r_f\mathbf{1}]^\top \mathbf{\Omega}_{a,t}^{-1}\mathbf{1}}$$

denotes the random return on the market portfolio of the risky assets, whereas the "aggregate" beta coefficients are given by

$$\boldsymbol{\beta}_{a,mt} = \frac{[E_{a,t}(\widetilde{\mathbf{r}}_{t+1}) - r_f\mathbf{1}]^\top \mathbf{\Omega}_{a,t}^{-1}\mathbf{1}}{[E_{a,t}(\widetilde{\mathbf{r}}_{t+1}) - r_f\mathbf{1}]^\top \mathbf{\Omega}_{a,t}^{-1}[E_{a,t}(\widetilde{\mathbf{r}}_{t+1}) - r_f\mathbf{1}]}[E_{a,t}(\widetilde{\mathbf{r}}_{t+1}) - r_f\mathbf{1}]$$

Note that the "aggregate" betas are time varying due to time varying beliefs about both the second moment and the first moment of the returns distribution.

**Remark**

The dynamical system (37), (36) is not deterministic. However, assuming that dividends evolve over time as an i.i.d. process, it can be useful to write down the deterministic system obtained by replacing $\widetilde{\mathbf{d}}_t$ with $\overline{\mathbf{d}} := E(\widetilde{\mathbf{d}}_t)$. One can easily see from (37), (36) that in order the system to be at a steady state, the stationary prices and returns $\overline{\mathbf{p}}$ and $\mathbf{r}$ must satisfy

$$\overline{\mathbf{p}} = \mathbf{S}^{-1}\sum_{h\in H} n_h\theta_h^{-1}\overline{\mathbf{\Omega}}_h^{-1}[\overline{\mathbf{f}}_h - r_f\mathbf{1}] \tag{38}$$

where $\overline{\mathbf{\Omega}}_h := \mathbf{\Omega}_h(\overline{\mathbf{r}}, \overline{\mathbf{r}}, ...)$, $\overline{\mathbf{f}}_h := \mathbf{f}_h(\overline{\mathbf{r}}, \overline{\mathbf{r}}, ...)$, and

$$\overline{\mathbf{r}} = \overline{\mathbf{P}}^{-1}\overline{\mathbf{d}} \tag{39}$$

where $\overline{\mathbf{P}} := diag(\overline{p}_1, \overline{p}_2, ..., \overline{p}_N)$ (note that $\overline{\mathbf{P}}^{-1}\overline{\mathbf{p}} = \mathbf{1}$). Equation (39) can be rewritten in the equivalent form

$$\overline{\mathbf{p}} = \overline{\mathbf{R}}^{-1}\overline{\mathbf{d}} \tag{40}$$

where $\overline{\mathbf{R}} := diag(\overline{r}_1, \overline{r}_2, ..., \overline{r}_N)$, or component by component as

$$\overline{p}_j = \frac{\overline{d}_j}{\overline{r}_j} \qquad j = 1, 2, ..., N$$

which gives the equilibrium prices through the usual discounted dividend formula via the appropriate rates of returns for each asset. Substitution of (40) into (38) yields

$$\overline{\mathbf{R}}^{-1}\overline{\mathbf{d}} = \mathbf{S}^{-1}\sum_{h\in H} n_h\theta_h^{-1}\overline{\mathbf{\Omega}}_h^{-1}[\overline{\mathbf{f}}_h - r_f\mathbf{1}]$$

where

$$\overline{\mathbf{\Omega}}_h := \mathbf{\Omega}_h(\overline{\mathbf{r}}, \overline{\mathbf{r}}, ...), \quad \overline{\mathbf{f}}_h := \mathbf{f}_h(\overline{\mathbf{r}}, \overline{\mathbf{r}}, ...)$$

i.e. a system of equilibrium equations in the returns $\overline{r}_1, \overline{r}_2, ..., \overline{r}_N$, from which in general the stationary returns (and then prices) can be solved for.

# 6 An example

In this section we provide a specific example of interaction of different beliefs types and analyze the resulting dynamics for market returns and aggregate beta coefficients. This example, which is similar to Chiarella Dieci and He (2006) considers two types of agents, *fundamentalists*, who have some information on the 'fundamentals' of the risky asset and who believe that prices will be driven back to fundamentals in the future, and *trend followers*, who may have no information on the fundamentals and who extrapolate the past returns into future returns. These two types of agents are the most common and popular ones in the literature on heterogeneous agent based models. In addition, we consider a third type of agents - *noise traders* - whose demand for each risky asset is treated as an exogenous random disturbance, described by an i.i.d. process with zero mean.

## 6.1 Fundamentalists and trend followers

The example follows the heterogeneous agent model with multiple risky assets studied in Chiarella Dieci and He (2006) and is related to earlier work by Chiarella and He, 2001b. We consider two types of agents, *fundamentalists* and *trend followers*.

Fundamentalists expect the future asset returns will be driven back towards some exogenously given levels $\boldsymbol{\rho} = [\rho_1, \rho_2, ..., \rho_N]^\top$, which are assumed to depend on 'fundamental' variables. This can be expressed as

$$E_{f,t}(\widetilde{\mathbf{r}}_{t+1}) = \mathbf{r}_{t-1} + \alpha(\boldsymbol{\rho} - \mathbf{r}_{t-1})$$

where $\alpha$ represents the expected speed of mean reversion. For the sake of simplicity, we assume that the fundamentalists have constant beliefs about the variance/covariance structure of returns determined - for instance - by the variance/covariance of the dividends

$$\boldsymbol{\Omega}_{f,t} = \overline{\boldsymbol{\Omega}}_f$$

The trend followers are assumed to compute the expected returns according to

$$E_{c,t}(\widetilde{\mathbf{r}}_{t+1}) = \mathbf{r}_{t-1} + \gamma(\mathbf{r}_{t-1} - \mathbf{u}_{t-1})$$

where $\gamma \geq 0$ is the extrapolation parameter and $\mathbf{u}_{t-1}$ is a vector of sample means of past realized returns $\mathbf{r}_{t-1}, \mathbf{r}_{t-2}, ....$ Similarly to Chiarella Dieci and He (2006), we assume that $\mathbf{u}_{t-1}$ is computed recursively as

$$\mathbf{u}_{t-1} = \delta \mathbf{u}_{t-2} + (1-\delta)\mathbf{r}_{t-1} \tag{41}$$

The variance/covariance matrix $\boldsymbol{\Omega}_{c,t}$ is assumed to consist of a constant component $\overline{\boldsymbol{\Omega}}_c$, and by a time-varying component $\lambda \mathbf{V}_{t-1}$, $\lambda \geq 0$.

$$\boldsymbol{\Omega}_{c,t} = \overline{\boldsymbol{\Omega}}_c + \lambda \mathbf{V}_{t-1}$$

where $\mathbf{V}_{t-1}$ is updated recursively as a function of past deviations from sample average returns, as follows

$$\mathbf{V}_{t-1} = \delta \mathbf{V}_{t-2} + \delta(1-\delta)(\mathbf{r}_{t-1} - \mathbf{u}_{t-2})(\mathbf{r}_{t-1} - \mathbf{u}_{t-2})^{\top} \tag{42}$$

Note that (41) and (42) can be considered as limiting cases of geometric decay processes, when the memory lag length tends to infinity.

We denote by $\theta_f$ and $\theta_c$ the risk aversion coefficients of the two agent-types, by $n_f$ and $n_c = 1 - n_f$ their market fractions, and by $\theta_a = \left(n_f \theta_f^{-1} + n_c \theta_c^{-1}\right)^{-1}$ the average risk aversion. Then the variances/covariances and expected excess returns are given, respectively, by

$$\mathbf{\Omega}_{a,t} = \theta_a^{-1} \left( \frac{n_f}{\theta_f} \overline{\mathbf{\Omega}}_f^{-1} + \frac{n_c}{\theta_c} \mathbf{\Omega}_{c,t}^{-1} \right)^{-1} = \left( \frac{n_f}{\theta_f} + \frac{n_c}{\theta_c} \right) \left( \frac{n_f}{\theta_f} \overline{\mathbf{\Omega}}_f^{-1} + \frac{n_c}{\theta_c} \mathbf{\Omega}_{c,t}^{-1} \right)^{-1}$$

$$\begin{aligned} E_{a,t}(\widetilde{\mathbf{r}}_{t+1}) &= \theta_a \mathbf{\Omega}_{a,t} \left[ \frac{n_f}{\theta_f} \overline{\mathbf{\Omega}}_f^{-1} E_{f,t}(\widetilde{\mathbf{r}}_{t+1}) + \frac{n_c}{\theta_c} \mathbf{\Omega}_{c,t}^{-1} E_{c,t}(\widetilde{\mathbf{r}}_{t+1}) \right] = \\ & \mathbf{r}_{t-1} + \left( \frac{n_f}{\theta_f} \overline{\mathbf{\Omega}}_f^{-1} + \frac{n_c}{\theta_c} \mathbf{\Omega}_{c,t}^{-1} \right)^{-1} \left[ \frac{\alpha n_f}{\theta_f} \overline{\mathbf{\Omega}}_f^{-1} (\boldsymbol{\rho} - \mathbf{r}_{t-1}) + \frac{\gamma n_c}{\theta_c} \mathbf{\Omega}_{c,t}^{-1} (\mathbf{r}_{t-1} - \mathbf{u}_{t-1}) \right] \end{aligned}$$

The general dynamic model given by (36) and (37) thus specializes to the following noisy nonlinear dynamical system

$$\mathbf{p}_t = \mathbf{S}^{-1} \left[ \left( \frac{n_f}{\theta_f} \overline{\mathbf{\Omega}}_f^{-1} + \frac{n_c}{\theta_c} \mathbf{\Omega}_{c,t}^{-1} \right) (\mathbf{r}_{t-1} - r_f \mathbf{1}) + \frac{\alpha n_f}{\theta_f} \overline{\mathbf{\Omega}}_f^{-1} (\boldsymbol{\rho} - \mathbf{r}_{t-1}) + \frac{\gamma n_c}{\theta_c} \mathbf{\Omega}_{c,t}^{-1} (\mathbf{r}_{t-1} - \mathbf{u}_{t-1}) \right]$$

$$\mathbf{r}_t = \mathbf{P}_{t-1}^{-1} (\mathbf{p}_t + \widetilde{\mathbf{d}}_t) - \mathbf{1}$$

where

$$\mathbf{\Omega}_{c,t} = \overline{\mathbf{\Omega}}_c + \lambda \mathbf{V}_{t-1}$$

and $\mathbf{u}_{t-1}$ and $\mathbf{V}_{t-1}$ are updated as follows

$$\mathbf{u}_t = \delta \mathbf{u}_{t-1} + (1-\delta) \mathbf{r}_t$$

$$\mathbf{V}_t = \delta \mathbf{V}_{t-1} + \delta(1-\delta)(\mathbf{r}_t - \mathbf{u}_{t-1})(\mathbf{r}_t - \mathbf{u}_{t-1})^{\top}$$

## 6.2 The effect of noise traders

The demand for the risky assets (number of shares) from the noise traders at time $t$ is described by the random vector $\widetilde{\boldsymbol{\xi}}_t := [\widetilde{\xi}_{1,t}, \widetilde{\xi}_{2,t}, ..., \widetilde{\xi}_{N,t}]^{\top}$, where the $\widetilde{\xi}_{j,t}$ are assumed i.i.d. with $E(\widetilde{\xi}_{j,t}) = 0$, $j = 1, 2, ..., N$. We also assume, for the sake of simplicity, that the standard deviation of the noise traders demand for each asset is proportional to the supply of the same asset in the market, i.e. $Var(\widetilde{\xi}_{j,t}) = q^2 s_j^2$, while demands for different assets are not correlated, $E(\widetilde{\xi}_{j,t}, \widetilde{\xi}_{k,t}) = 0$, $j, k = 1, 2, ..., N$.

Set $\widetilde{\boldsymbol{\Xi}}_t := diag(\widetilde{\xi}_{1,t}, \widetilde{\xi}_{2,t}, ..., \widetilde{\xi}_{N,t})$. The market clearing conditions in the presence of noise traders thus become

$$\theta_a^{-1}\boldsymbol{\Omega}_{a,t}^{-1}[E_{a,t}(\widetilde{\mathbf{r}}_{t+1}) - r_f\mathbf{1}] + \widetilde{\boldsymbol{\Xi}}_t\mathbf{p}_t = \mathbf{S}\mathbf{p}_t$$

and the market clearing prices thus become

$$\mathbf{p}_t = (\mathbf{S} - \widetilde{\boldsymbol{\Xi}}_t)^{-1}\theta_a^{-1}\boldsymbol{\Omega}_{a,t}^{-1}[E_{a,t}(\widetilde{\mathbf{r}}_{t+1}) - r_f\mathbf{1}]$$

where $E_{a,t}(\widetilde{\mathbf{r}}_{t+1})$ and $\boldsymbol{\Omega}_{a,t}$ are as in the previous section. Note that the introduction of noise traders is formally equivalent to assuming a noisy supply vector $\widetilde{\mathbf{s}}_t = \mathbf{s} - \widetilde{\boldsymbol{\xi}}_t$.

### 6.3 Simulation results

...

## 7 Discussion

...

## References


Böhm, V., Chiarella, C., 2005. Mean variance preferences, expectations formation and the dynamics of random asset prices. Mathematical Finance 15(1), 61-97.

Brock, W.A., Hommes, C.H., 1998. Heterogeneous beliefs and routes to chaos in a simple asset pricing model. Journal of Economic Dynamics and Control 22, 1235-1274.

Chiarella, C., He, X.-Z., 2001a. Heterogeneous beliefs, risk and learning in a simple asset pricing model. Computational Economics 19 (1), 95-132.

Chiarella, C., He X.-Z., 2001b. Asset price and wealth dynamics under heterogeneous expectations. Quantitative Finance 1(5), 509-526.

Chiarella, C., Dieci, R., He, X.-Z., 2006. Heterogeneous expectations and speculative behavior in a dynamic multi-asset framework. Journal of Economic Behavior and Organization, in press.

Huang, C., Litzenberger R.H., 1988. Foundations for financial economics. North-Holland, Amsterdam.

Lintner, J., 1969. The aggregation of investor's diverse judgements and preferences in purely competitive security markets. Journal of Financial and Quantitative Analysis 4 (4), 347-400.