ZERO-BETA CAPM UNDER HETEROGENEOUS BELIEFS

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ABSTRACT. We provide a direct extension to the mean-variance equilibrium model under heterogeneous beliefs considered by Chiarella, Dieci and He (2006*a*,2006*b*) by removing the assumption of the existence of a risk-free asst in the market. Market aggregation and equilibrium prices are derived using a consensus belief, also the zero-beta CAPM is derived under the consensus belief and we show that the zero-beta rate is actually the aggregate marginal certainty equivalent wealth referred to as the shadow price. Various special cases of market equilibrium are considered in which the impact of heterogeneity from different sources are examined and compared with the case when risk-free asset exists. We found that our result support the argument in Miller (1977) that "divergence of opinion corresponds to lower future asset returns" and the subsequent empirical findings by Deither et al (2002) under some conditions. Finally, Markowitz's portfolio frontiers under heterogeneous beliefs are discussed and a simple numerical example is given.

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1. INTRODUCTION

The Capital Asset Pricing Model developed simultaneously and independently by Sharp (1964), Lintner (1965) and Mossin (1966) is perhaps the most influential object in modern finance. It provides the theoretical foundation for relating risks linearly with expected return of a asset. However, CAPM is not without criticism, it assumes (1) all investors make portfolio selection decisions according to the criteria set out in Markowitz (1959) within the mean-variance framework, (2) investors have homogeneous beliefs about the future state of the market, (3) unlimited borrowing and lending at a single risk-free rate, (4) a frictionless market. Assumptions (2) and (3) might be the most restrictive of them all, since we know that they are far from being realistic. This paper seeks to generalize the CAPM by removing assumptions (2) and (3), find out the consequences of such generalization and thus study the impact of heterogeneity in beliefs as well as the lacking of a risk-free asset in the market.

Many literatures made significant contribution to the understanding of the impact of heterogeneous beliefs amongst investors on market equilibrium. Some have considered the problem in discrete time, (for example, see Lintnter (1969), Rubinstein (1976), Fan (2003), Sun and Yang (2003), Chiarella, Deici and He (2006*a*,2006*b*), Jouni and Napp (2006) and Sharpe (2007)) and others in continuous time (for example, see Williams (1977), Detemple and Murthy (1994), Zapatero (1998) and Jouni and Napp (2004)). Equilibrium models have been developed to consider the impact of heterogeneity, either in the mean-variance framework (see, Lintner (1969), William (1977) and Sun and Yang (2003)) or under the Arrow-Debreu contingent claims economy (see, Rubinstein (1976), Abel (1989,2002)). Heterogeneity may be caused by difference in information or difference in opinion. If caused by the former, then investors may update their beliefs as new information become available, Bayesian updating rule is often used (see, for example, Williams (1977) and Zapatero (1998)). If cause by the latter, then investor may revise their portfolio strategies as their views of the market change over time (see, for example Lintner (1969) and Rubinstein (1975)). The actual cause of heterogeneity is irrelevant in our case since we only consider a two-period economy, however it may become important in dynamic models.

Amongst all the literatures mentioned above, only Lintner (1969) and Sun and Yang (2003) considered market equilibrium and asset prices without both assumptions (2) and (3) of the traditional CAPM. Lintner (1969) and Black (1972) were the first papers to consider the problem of market equilibrium without assumption (2), however Lintner did not put too much weight on it and thought it as just a mere extension of the case with a risk-free asset and did not have a detailed discussion, while Black was really the one to extend the CAPM by removing assumption (2) and thus developed the zero-beta CAPM, it provided some theoretical explanations on the early empirical tests of CAPM, especially why low beta stocks lie above the *Security Market Line* (SML) and high beta stocks lie below it while according to CAPM they should all lie exactly on the SML, Black argued using the zero-beta CAPM that the SML consists of two line segments of different slopes rather than a single straight line and first line segment has a higher slope than the second, which partially explains the empirical findings. The question is does the zero-beta CAPM still exist without assumption (3), the answer is yes and it was proved by a recent paper Sun and Yang (2003), although is somewhat mathematically demanding, but it proved both the existence of market equilibrium price and the zero-beta CAPM under heterogeneous beliefs in a rather general setting within the mean-variance framework. However, the mere proof of existence is not enough, one needs to use the resulting model to give more insights into our financial market and provide theoretical explanations of the empirical phenomenons found by other researchers, to do so, we may need to sacrifice some generality, but for a worthy course.

In this paper, we consider a market without assumptions (2) and (3), but we assume that investors all have *Constant Absolute Risk Aversions* (CARA) type utility functions. Heterogeneity is introduced by allowing investors to have different probability assessments about expected end-of-period asset payoffs and variance/covariance of asset payoffs and also investors have different attitude towards risk represented by their *Absolute Risk Aversion Coefficient*. The objective of this paper is to

1 . aggregate the market through the construction of a consensus belief, which if undertaken by every investor would generate the same equilibrium prices as the heterogeneous market, also provide some economical intuition for the resulting consensus belief
2 . prove the zero-beta CAPM under the consensus belief, give economical interpretation for zero-beta rate and risk-premium implied by the model.
3 . consider some special cases of market equilibrium attempting to assess the impact specific heterogeneity and also the impact of the existence of a risk-free asset on the market equilibrium prices, provide some theoretical explanations for certain financial anomalies.
4 . study the impact of heterogeneity on the market's portfolio frontiers, compare the frontiers with and without a risk-free asset, examine the validity of the *Two Fund Separation Theorem* and some standard features of the portfolio frontiers under heterogeneous beliefs.

We solve individual investor's optimal portfolio selection problem by Lagrangian multipliers and found similar economical interpretation for these values to Lintner (1969)and refer to them as investors' shadow prices, which measures their marginal certain equivalent wealth. The consensus belief is a weighted average of individual investors' risk-aversions, var/cov matrices, expected payoffs and also shadow prices without the assumption of a risk-free asset, a very intuitive result. We show that zero-beta CAPM holds under the consensus belief, the zero-beta rate is actually the aggregated shadow price, and the risk-premium is proportional to aggregated risk-aversion, average market wealth and the market's aggregated volatility which is consistent with standard finance theory. Moreover, we provide some theoretical evidence which support Miller's argument that ”greater divergence of opinions among investor leads to lower future returns” by applying a *Mean Preserved Spread* (MPS) to the aggregated var/cov matrix and the aggregate expected returns vector. We also shed some light on the risk-premium puzzle by comparing the average and aggregated var/cov matrices, we prove in the two-investors case that aggregate var/cov matrix produces less portfolio risk than the average one. Lastly, we show by a simple numerical example that *Two Fund Separation* and the tangency relation generally do not hold under heterogeneous belief, furthermore, market portfolio is always mean-variance efficient under the consensus belief while individual investors optimal portfolios might not be efficient, this is also consistent with the simulated results of Sharpe (2007).

The paper is structured as follows, section 2 provides a generalization of the static mean-variance model developed by Chiarella et al(2006*a*) by removing the assumption of a risk-free asset. Section 3 shows the construction of a consensus belief linking the heterogeneous market with an equivalent homogeneous market. Section 4 proves the heterogeneous zero-beta CAPM using the consensus belief. Section 5 examines the impact of heterogenous belief and the existence of the risk-free asset have on the market by considering some special cases of market equilibrium. Section 6 compare and contrast the portfolio frontiers under heterogeneous beliefs with the standard case. Section 7 provides another method for market aggregation by allowing investors to form their beliefs in terms of asset returns rather than asset payoffs and Section 8 concludes.

2. Mean-Variance Analysis under Heterogeneous Beliefs without a Riskless Asset

This section is trying to extend the static mean-variance model developed in section 2 of Chiarella, Dieci and He (2006a), by removing the assumption of the existence of a risk-free asset, since in reality, there is not a single security that is risk-free, even bonds have uncertainty associated with their returns and this uncertainty becomes greater as the time to maturity of the bonds increases. Therefore it would be ideal to have a static mean-variance model in a market without access to a risk-free asset.

Consider a market with N risky assets, indexed by $j, k = 1, 2, \cdots, N$ and I investors indexed by $i = 1, 2, \cdots, I$. We assume that each investor has his/her own set of beliefs about the market in terms of means, variances and covariances of the payoffs of the assets, denoted by

$$y_{i,j} = E_i[\tilde{x}_j], \quad \sigma_{i,jk} = Cov_i(\tilde{x}_j, \tilde{x}_k) \tag{2.1}$$

where $\tilde{x_j}, \tilde{x_k}$ are the random payoff of asset j and k respectively.

Let $\tilde{\mathbf{x}} = (\tilde{x_1}, \cdots, \tilde{x_N})^T$ be the payoff vector of risky assets, then we can define the mean vector and variance/covariance matrix of the payoffs of N assets as follows,

$$\mathbf{y}_i = \mathbb{E}_i(\tilde{\mathbf{x}}) = (y_{i,1}, y_{i,2}, \cdots, y_{i,N})^T, \qquad \Omega_i = (\sigma_{i,jk})_{N\times N}$$

Denote $\mathcal{B}_i = (\mathbb{E}_i(\tilde{\mathbf{x}}), \Omega_i)$ as the set of subjective beliefs of investor i, and let

$$\mathbf{z}_i = (z_{i,1}, z_{i,2}, \cdots, z_{i,N})^T$$

be the investment of investor i in the risky assets in absolute amount (number of shares), and W_0^i be the initial wealth of investor i in dollar amount. Then the end-of-period wealth of this portfolio of investor i is simply

$$\tilde{W}_i = \tilde{\mathbf{x}}^T \mathbf{z}_i.$$

The mean and variance of $\tilde{W}_i$ are given respectively, by

$$\mathbb{E}_i(\tilde{W}_i) = \mathbf{y}_i^T \mathbf{z}_i, \qquad \sigma_i^2(\tilde{W}_i) = \mathbf{z}_i^T \Omega_i \mathbf{z}_i. \tag{2.2}$$

Now we must make certain assumptions of each investor's utility function.[1]

(H1) *Assume investor i's utility function is the constant absolute risk aversion (CARA) utility function $U_i(w) = -e^{-\theta_i w}$ for all i.*

(H2) *Assume investor i's end-of-period wealth $\tilde{W}_i$ follows a normal distribution for all i.*

Under (H1) and (H2), maximizing investor i's expected utility is equivalent to maximizing his/her certainty equivalent end-of-period wealth, given by $\mathbf{y}_i^T \mathbf{z}_i - \frac{\theta_i}{2} \mathbf{z}_i \Omega_i \mathbf{z}_i$ where θ_i is the absolute risk aversion of investor i.

Given (H1) and (H2), the optimal portfolio of investor i in absolute amount (ie. $\mathbf{z}_i$) is determined by

$$\max_{\mathbf{z}_i} \left[\mathbf{y}_i^T \mathbf{z}_i - \frac{\theta_i}{2} \mathbf{z}_i \Omega_i \mathbf{z}_i \right]$$

subject to the wealth constraint

$$\mathbf{p}_0^T \mathbf{z}_i = W_0^i. \tag{2.3}$$

[1]The assumptions made here are standard in the mean-variance framework.

Lemma 2.1. *Under assumptions (H1) and (H2), the optimal risky portfolio $\mathbf{z}_i^*$ of investor i at the market equilibrium is given by*

$$\mathbf{z}_i^* = \theta_i^{-1}\Omega_i^{-1}[\mathbf{y}_i - \lambda_i^*\mathbf{p}_0]. \tag{2.4}$$

where

$$\lambda_i^* = \frac{\mathbf{p}_0^T\Omega_i^{-1}\mathbf{y}_i - \theta_i W_0^i}{\mathbf{p}_0^T\Omega_i^{-1}\mathbf{p}_0} \tag{2.5}$$

Proof. Let λ_i be the Lagrange multiplier and set

$$L(\mathbf{z}_i, \lambda_i) := \mathbf{y}_i^T\mathbf{z}_i - \frac{\theta_i}{2}\mathbf{z}_i\Omega_i\mathbf{z}_i + \lambda_i[\mathbf{p}_0^T\mathbf{z}_i - W_0^i] \tag{2.6}$$

Then the optimal portfolio of agent i is determined by the first order conditions since $U_i(.)$ is concave,

$$\frac{\partial L}{\partial \mathbf{z}_i} = \mathbf{0} \qquad \Rightarrow \qquad \mathbf{z}_i = \theta_i^{-1}\Omega_i^{-1}[\mathbf{y}_i - \lambda_i\mathbf{p}_0]. \tag{2.7}$$

Substituting (2.7) into (2.3) yields (2.5), this completes the proof. □

Lemma 2.1 shows that the optimal demand of investor-i is determined by his/her risk aversion θ_i and his/her belief about the expected payoffs and variance/covariance matrix of the risky assets' payoffs, however, now without access to a risk-free asset, it also depends on λ_i^*, which is determined by investor i's risk aversion, belief about expected payoffs and variance/covariance matrix of the payoffs as well as the price of the assets. As it was already mentioned in Lintner's earlier work (Lintner (1969)) that λ_i^* is a shadow price, which actually measures the *marginal real (riskless) certainty-equivalent of investor i's end-of-period wealth.* To show this, re-write equation (2.6) as

$$L(\mathbf{z}_i, \lambda_i) := Q_i(\mathbf{z}) + \lambda_i[\mathbf{p}_0^T\mathbf{z}_i - W_0^i], \tag{2.8}$$

where

$$Q_i(\mathbf{z}) := \mathbf{y}_i^T \mathbf{z}_i - \frac{\theta_i}{2} \mathbf{z}_i \Omega_i \mathbf{z}_i \tag{2.9}$$

is the certainty equivalent end-of-period wealth of investor i, this definition is also used in Lintner (1969). Differentiating equation (2.8) w.r.t $\mathbf{z}_i$ and apply the first order condition, one can obtain

$$\frac{\partial Q_i(\mathbf{z}^*)}{\partial \mathbf{z}_i} = \lambda_i^* \mathbf{p}_0$$

which leads to

$$\lambda_i^* = \frac{1}{p_{0j}} \frac{\partial Q_i(\mathbf{z}^*)}{\partial z_{ij}} \qquad \forall j. \tag{2.10}$$

Equation (2.10) indicates that λ_i^*, to be precise, actually measures *investor i's optimal marginal certainty equivalent end-of-period wealth per unit of asset j relative to its price for at market equilibrium* and it is constant across all assets. Lintner (1969) referred to this measurement as the *shadow price*, this paper is going to adopt this terminology and call it the shadow price of investor i. In general, the shadow price is not necessary the same for every investor, however, it becomes the same when there exist a risk-free asset in the market, because the existence of a risk-free asset (f) implies that equation (2.10) will become

$$\lambda_i^* = \frac{1}{p_{0j}} \frac{\partial Q_i(\mathbf{z}^*)}{\partial z_{i0}} \tag{2.11}$$

where z_{i0} denotes the absolute investment in the risk-free asset. Let the current price of f be 1 and its payoff be $R_f = 1 + r_f$, which means that $\lambda_i^* = R_f$ for all investor i. Therefore everyone's shadow price is equal to the payoff of the risk-free asset.

3. Consensus Belief and Equilibrium Asset Prices

In this section, we will show that a consensus belief exists in our defined market, and then use the consensus belief to derive a set market equilibrium asset prices. Definition of a consensus belief is given in Definition 3.1.

Market equilibrium asset prices is a vector of asset prices $\mathbf{p}_o$ under which each individual optimal demands (2.4) are satisfied and the market aggregation condition

$$\sum_{i=1}^{I} \mathbf{z}_i^* = \mathbf{z}_m, \tag{3.1}$$

also hold. Using the aggregation condition (3.1), $\mathbf{p}_0$ can be found in terms of each investor's subjective beliefs, given by

$$\mathbf{p}_0 = \left(\sum_{i=1}^{I} \theta_i^{-1} \lambda_i^* \Omega_i^{-1} \right)^{-1} \left[\left(\sum_{i=1}^{I} \theta_i^{-1} \Omega_i^{-1} \mathbf{y}_i \right) - \mathbf{z}_m \right] \tag{3.2}$$

[2]

Definition 3.1. *A belief $\mathcal{B}_a = (\mathbb{E}_a(\tilde{\mathbf{x}}), \Omega_a)$, defined by the expected payoff of the risky assets $\mathbb{E}_a(\tilde{\mathbf{x}})$ and the variance and covariance matrix of the risky asset payoffs Ω_a, is called a* **consensus belief** *if and only if the equilibrium price under the heterogeneous beliefs is also the equilibrium price under the homogeneous belief $\mathcal{B}_a$.*

We now try to construct a consensus belief, from which market equilibrium prices, $\mathbf{p}_0$ can be determined in terms of the consensus belief constructed. [3]

[2] One might realized that the expression above is implicit since λ_i^* depends on $\mathbf{p}_0$ and question the uniqueness of the equilibrium price vector, however according to our numerical simulations, the price vector in (3.2) appears to be unique, which implies that λ_i^* is also unique for all i

[3] If we assume that the equilibrium price vector is the previous section is unique in our market, then the consensus belief constructed in this section is also uniquely defined.

Proposition 3.2. *Under assumptions (H1) and (H2), let*

$$\Theta := \left(\frac{1}{I}\sum_{i=1}^{I}\theta_i^{-1}\right)^{-1}, \tag{3.3}$$

$$\lambda_a := \frac{1}{I}\Theta\sum_{i=1}^{I}\theta_i^{-1}\lambda_i^*. \tag{3.4}$$

Then

(i) *the unique consensus belief $\mathcal{B}_a$ is given by*

$$\Omega_a = \Theta^{-1}\lambda_a\left(\frac{1}{I}\sum_{i=1}^{I}\lambda_i^*\theta_i^{-1}\Omega_i^{-1}\right)^{-1}, \tag{3.5}$$

$$\mathbf{y}_a = \mathbb{E}_a(\tilde{\mathbf{x}}) = \Theta\Omega_a\left(\frac{1}{I}\sum_{i=1}^{I}\theta_i^{-1}\Omega_i^{-1}\mathbb{E}_i(\tilde{\mathbf{x}})\right); \tag{3.6}$$

(ii) *the market equilibrium price $\mathbf{p}_o$ is determined by*

$$\mathbf{p}_0 = \frac{1}{\lambda_a}\left[\mathbf{y}_a - \frac{1}{I}\Theta\Omega_a\mathbf{z}_m\right]; \tag{3.7}$$

(iii) *the equilibrium optimal portfolio of agent i is given by*

$$\mathbf{z}_i^* = \theta_i^{-1}\Omega_i^{-1}\left[(\mathbf{y}_i - \frac{\lambda_i^*}{\lambda_a}\mathbf{y}_a) + \frac{\lambda_i^*}{I\lambda_a}\Theta\Omega_a\mathbf{z}_m\right]. \tag{3.8}$$

Proof. On the one hand, from Definition 3.1, if the consensus belief $\mathcal{B}_a = (\mathbb{E}_a(\tilde{\mathbf{x}}), \Omega_a)$ exists, then under the homogenous belief, it must be true that

$$\mathbf{z}_i^* = \Theta^{-1}\Omega_a^{-1}[\mathbf{y}_a - \lambda_a^*\mathbf{p}_0]. \tag{3.9}$$

Applying the market equilibrium condition to (3.9), we must have

$$\mathbf{z}_m = \sum_{i=1}^{I}\mathbf{z}_i^* = I\left[\Theta^{-1}\Omega_a^{-1}[\mathbf{y}_a - \lambda_a\mathbf{p}_0]\right]. \tag{3.10}$$

This leads to the equilibrium price (3.7).

On the other hand, it follows from the individuals demand (2.4) and the market clearing condition (3.1) that, under heterogenous beliefs

$$\mathbf{z}_m = \sum_{i=1}^{I} \mathbf{z}_i^* = \sum_{i=1}^{I} \theta_i^{-1}\Omega_i^{-1}[\mathbf{y}_i - \lambda_i^* \mathbf{p}_0]. \tag{3.11}$$

Under the definitions (3.5) and (3.6), we can re-write equation (3.11) as

$$\begin{aligned}\mathbf{z}_m &= \sum_{i=1}^{I} \theta_i^{-1}\Omega_i^{-1}\mathbf{y}_i - \Big(\sum_{i=1}^{I} \theta_i^{-1}\lambda_i^*\Omega_i^{-1}\Big)\mathbf{p}_0 \\ &= I\Theta^{-1}\Omega_a^{-1}\mathbf{y}_a - I\Theta^{-1}\lambda_a\Omega_a^{-1}\mathbf{p}_0,\end{aligned} \tag{3.12}$$

which leads to the same market equilibrium price (3.7). This shows that $\mathcal{B}_a = \{\Omega_a, \mathbf{y}_a\}$ defined in (3.5) and (3.6) is the consensus belief. Inserting (3.7) into (2.4) will give the equilibrium optimal portfolio (3.8) of investor i. □

Proposition 3.2 shows the existence of a unique consensus belief in a market without a risk-free asset and how it can be constructed from heterogeneous beliefs. Proposition 3.2 provides the main result, which will be used in the following section to deduce a CAPM-like relation between risk and return.

Define the current wealth of the entire market as

$$W_{m0} := \mathbf{z}_m^T \mathbf{p}_0 = \sum_{i=1}^{I} W_0^i \tag{3.13}$$

From equation (3.13), it can be seen that

$$W_{m0} = \lambda_a \mathbf{z}_m^T(\mathbf{y}_a - \Theta\Omega_a\mathbf{z}_m/I)$$

from which we obtain

$$\lambda_a = \frac{\mathbf{z}_m^T\mathbf{y}_a - \Theta\mathbf{z}_m^T\Omega_a\mathbf{z}_m/I}{W_{m0}} \tag{3.14}$$

Expression in (3.14) will make the proof of Proposition 4.1 in the next section easier.

4. The CAPM without a risk-free asset under Heterogeneous Beliefs

It was seen in Chiarella, Deici and He (2006*a*) that standard CAPM relation can be developed under the consensus belief $\mathcal{B}_a$ when the beliefs are heterogeneous, known as the HCAPM. Now the question is that does a similar relation still exist in a market without a risk-free asset under heterogenous beliefs. In the homogeneous case, a CAPM-like relation was developed in Black (1972) and Lintner (1969), called *Zero-beta CAPM*. In this section, we show that such a relation between return and risk still exist in the heterogenous case using Proposition 3.2, which constitutes the second main set of results of this paper.

Let the future payoff of the entire market is given by $\tilde{W}_m = \tilde{\mathbf{x}}^T\mathbf{z}_m$ and its current market value is $W_{m0} = \mathbf{z}_m^T\mathbf{p}_0$. Hence under the consensus belief $\mathbb{B}_a$,

$$\mathbb{E}_a(\tilde{W}_m) = \mathbf{y}_a^T\mathbf{z}_m, \qquad \sigma^2_{a,m} = Var(\tilde{W}_m) = \mathbf{z}_m^T\Omega_a\mathbf{z}_m.$$

Define the returns

$$\tilde{r}_j = \frac{\tilde{x}_j}{p_{j,o}} - 1, \qquad \tilde{r}_m = \frac{\tilde{W}_m}{W_{m,o}} - 1$$

and set

$$\mathbb{E}_a(\tilde{r}_j) = \frac{\mathbb{E}_a(\tilde{x}_j)}{p_{j,o}} - 1, \qquad \mathbb{E}_a(\tilde{r}_m) = \frac{\mathbb{E}_a(\tilde{W}_m)}{W_{m,o}} - 1.$$

Proposition 4.1. *In equilibrium, the CAPM-like relation between expected return and risk under heterogeneous beliefs can be expressed as*

$$\mathbb{E}_a[\tilde{\mathbf{r}}] - (\lambda_a - 1)\mathbf{1} = \boldsymbol{\beta}[\mathbb{E}_a(\tilde{r}_m) - (\lambda_a - 1)], \tag{4.1}$$

where

$$\boldsymbol{\beta} = (\beta_1, \beta_2, \cdots, \beta_N)^T, \qquad \beta_j = \frac{Cov_a(\tilde{r}_m, \tilde{r}_j)}{\sigma_a^2(\tilde{r}_m)}, \qquad j = 1, \cdots, N,$$

and the mean and variance/covariance of returns under the consensus belief $\mathcal{B}_a$ are defined similarly, λ_a is given by equation (3.14).

Proof. The equilibrium price vector in (3.7) can be re-written to express the price of each asset

$$\begin{aligned} p_{0,j} &= \frac{1}{\lambda_a}(y_{a,j} - \Theta/I \sum_{k=1}^{N} \sigma_{j,k} z_{m,k}) \\ &= \frac{1}{\lambda_a}[y_{a,j} - \frac{\Theta}{I} Cov_a(\tilde{x}_j, \tilde{W}_m)]. \end{aligned} \tag{4.2}$$

It follows from (4.2) that

$$\begin{aligned} y_{a,j} - \lambda_a p_{0,j} &= \frac{\Theta}{I} Cov_a(\tilde{x}_j, \tilde{W}_m), \\ \frac{y_{a,j}}{p_{0,j}} - \lambda_a &= \frac{1}{p_{0,j}} \frac{\Theta}{I} Cov_a(\tilde{x}_j, \tilde{W}_m). \end{aligned}$$

Hence

$$\mathbb{E}_a(\tilde{r}_j) - (\lambda_a - 1) = \frac{1}{p_{0,j}} \frac{\Theta}{I} Cov_a(\tilde{x}_j, \tilde{W}_m) \tag{4.3}$$

Using the definition of λ_a in (3.14), we obtain

$$\mathbb{E}_a(\tilde{r}_m) - (\lambda_a - 1) = \frac{\mathbf{y}_a^T \mathbf{z}_m}{\mathbf{z}_m^T \mathbf{p}_0} - \lambda_a = \frac{\mathbf{y}_a^T \mathbf{z}_m}{W_{m0}} - \frac{\mathbf{z}_m^T \mathbf{y}_a - \Theta \mathbf{z}_m^T \Omega_a \mathbf{z}_m / I}{W_{m0}}.$$

Thus

$$\mathbb{E}_a(\tilde{r}_m) - (\lambda_a - 1) = \frac{\Theta \mathbf{z}_m^T \Omega_a \mathbf{z}_m / I}{W_{m0}} \neq 0. \tag{4.4}$$

Dividing (4.3) by (4.4) leads to

$$\frac{\mathbb{E}_a(\tilde{r}_j)-(\lambda_a-1)}{\mathbb{E}_a(\tilde{r}_m)-(\lambda_a-1)} = \frac{\left(\frac{1}{p_{0,j}}\frac{\Theta}{I}Cov_a(\tilde{x}_j,\tilde{W}_m)\right)}{\left(\frac{\Theta \mathbf{z}_m^T\Omega_a\mathbf{z}_m/I}{W_{m0}}\right)} = \frac{\frac{1}{p_{0,j}}Cov_a(\tilde{x}_j,\tilde{W}_m)}{\frac{\sigma^2_{a,m}}{W_{m0}}}$$

$$= \frac{Cov_a\left(\frac{\tilde{x}_j}{p_{0,j}},\frac{\tilde{W}_m}{W_{m0}}\right)}{\frac{\sigma^2_{a,m}}{W^2_{m0}}} = \frac{Cov_a(\tilde{r}_j,\tilde{r}_m)}{\sigma^2_a(\tilde{r}_m)} = \beta_j. \tag{4.5}$$

This lead to the CAPM-like relation in (4.1). □

The equilibrium relation (4.1) is the standard Zero-beta CAPM except that the mean and variance/covariance are calculated based on the consensus belief $\mathcal{B}_a$, it will be referred to as the *Zero-beta heterogeneous Capital Asset Pricing Model* (ZHCAPM). The zero-beta rate in this case is $\lambda_a - 1$, we can see this by re-writing equation (4.1) for asset j

$$\mathbb{E}_a[\tilde{r}_j]-(\lambda_a-1) = \beta_j[\mathbb{E}_a(\tilde{r}_m)-(\lambda_a-1)]. \tag{4.6}$$

Clearly, when $\beta_j = 0$, $\mathbb{E}_a[\tilde{r}_j] = \lambda_a - 1$. This indicates that the portfolio which has a zero beta coefficient, that is the zero-beta portfolio of the market portfolio, its aggregate expected return must be $\lambda_a - 1$, where λ_a is the aggregate shadow price.

Furthermore, we can observe the risk premium in equation (4.4), which in this case is the difference between the aggregate market return and the zero-beta rate. (4.4) shows that the risk premium is given by the expression $\frac{\Theta \mathbf{z}_m^T\Omega_a\mathbf{z}_m/I}{W_{m0}}$. In principle, the risk premium should be proportional to the aggregated ARR, the average wealth level and the aggregate volatility in the market, which can seen, are fully captured in our model. According to the expression given above, the risk premium in our model is positively related to the ARR and the aggregate market volatility and inversely related to the average wealth level in the market. This seems to be an intuitive result, for which many econometricians attempted to provide some theoretical explanations, our model shows that it is simply result of utility maximization in a mean-variance type of market driven by heterogeneous investors.

5. Aggregation and Equilibrium Prices under Some Special Cases

In this section, we will analyze the equilibrium prices of the market under some special cases in which we have homogeneity in some aspects aiming to link our model to the one in which there exists a risk-free asset. We will also consider the effect of a *Mean Preserved Spread* on some of the aggregated parameters of the market and the impact it has on the equilibrium prices and expected returns.

5.1. **Effect of Shadow Prices on the Equilibrium Prices.** As discussed in section 2, investor i's shadow price is given by equation (2.10), which becomes constant across all investor when there exists a risk-free asset in the market[4]. In fact, we have $\lambda_i^* = R_f$ for all i, which according the definition in (3.4) implies that $\lambda_a = R_f$, this means that the aggregate shadow price in the market is equal to the payoff of the risk-free asset with the assumption of the existence of a risk-free asset. Substituting $\lambda_i^* = \lambda_a = R_f$ into equation (3.5), (3.6), (3.7) and (3.8) will lead to the consensus variance/covariance matrix, consensus expected payoff vector, equilibrium price vector and the equilibrium optimal demand of agent i respectively. This means the static equilibrium model developed in this paper is consistent with the one constructed in Chiarella, Deici and He (2006*a*) except that we do not require the existence of a risk-free asset as an assumption.

It is also important to examine the relationship between individual shadow prices and

[4]The assumption that $\lambda_i^* = \lambda$ is quite absurd without the existence of a risk-free asset, since the shadow price differs across individual investors even when investors have the same attitude towards risk and their subjective beliefs of the market are identical.

the aggregate shadow price. To observe this relationship, let

$$\lambda_a := \lambda_a(\lambda_1^*, \lambda_2^*, \cdots, \lambda_I^*, \theta_1, \theta_2, \cdots, \theta_I),$$

the we calculate the first partial derivative of λ_a w.r.t λ_i^*,

$$\frac{\partial \lambda_a}{\partial \lambda_i^*} = \frac{\Theta \theta_i^{-1}}{I} > 0 \tag{5.1}$$

equation (5.1) shows that investor i's shadow price has a positive effect on the aggregate shadow price, and the rate of increase depends on θ_i, the ARR of investor i, to see the exact relation, we can take the second partial derivative w.r.t θ_i,

$$\frac{\partial^2 \lambda_a}{\partial \lambda_i^* \partial \theta_i} = \frac{1}{I}\theta_i^{-3}\Theta(\frac{1}{I}\Theta - \theta_i) \tag{5.2}$$

From equation (5.2), it is clear that if $\theta_i \geq \frac{1}{I}\Theta$, then $\frac{\partial^2 \lambda_a}{\partial \lambda_i^* \partial \theta_i} \leq 0$ and vice-versa. In reality, the number of investors I will be large, therefore it is unlikely to have $\theta_i < \frac{1}{I}\Theta$, hence the rate of increase in (5.2) should be negative in any reasonable case. In other words, increase in a less risk averse investor's shadow price will result in a sharper increase in the aggregate shadow price than an increase in a highly risk averse investor's shadow price.

Next, we want to see what kind of a role of initial wealth distribution play in determining the equilibrium prices. Fan (2003) proved the *Second Welfare Theorem* in a general two-period economy without specifying the type of utility function for any investors, the theorem states that investors with large capital endowments would have lower marginal utilities of capital endowments, whose utility is weighted more in the total market utility. In our case, if marginal utility for a individual investor is represented by his/her shadow price (λ_i^*), then from equation (2.5), it is clear that a large initial wealth or capital endowment would lead to a lower marginal utility. Also, if we take the expression of the equilibrium price vector in (3.2), it can be seen that (λ_i^*) is inversely related to the price vector for any i, this suggests an investor with a lower shadow price or marginal utility would have a stronger impact on the market prices of assets, which is consistent with the *Second Welfare Theorem*, so an investor with a larger capital is more influential in the market. However, *Second Welfare Theorem* is no longer true if we assume the existence of a risk-free asset in our market since in that case shadow prices or marginal utilities would be constant across all investor, hence the market prices are independent of the initial wealth distribution. This is intuitively correct, because if everyone can borrow unlimited capital without incurring any risk, then the initial wealth of investors becomes irrelevant in determining the equilibrium prices.

5.2. **Homogeneous Expected payoffs and Variance/covariance matrices.** In this case, we assume that investors' beliefs of the market are homogenous, that is

$$\Omega_i = \Omega, \qquad \mathbf{y}_i = \mathbf{y} \qquad \forall i \tag{5.3}$$

If we substitute (5.3) into (3.5) and (3.6) will yield the following result,

$$\Omega_a = \Omega \qquad \mathbf{y}_a = \mathbf{y} \qquad i.e. \qquad \mathcal{B}_i = \mathcal{B}_a \qquad \forall i \tag{5.4}$$

which is quite intuitive, since every investor's belief about the market are identical, then a consensus investor should have exactly the set of belief. Now let us look at the equilibrium price in this case, according to (5.4), the equilibrium price vector can be written as

$$\mathbf{p}_0 = \frac{1}{\lambda_a}\left[\mathbf{y} - \frac{1}{I}\Theta\Omega\mathbf{z}_m\right] \tag{5.5}$$

with

$$\lambda_a = \frac{\mathbf{z}_m^T\mathbf{y} - \Theta\mathbf{z}_m^T\Omega\mathbf{z}_m/I}{W_{m0}} \tag{5.6}$$

It can be seen from equation (5.5) together with (5.6) that the only source of heterogeneity is the risk aversion coefficient, prices are independent of the initial wealth distribution amongst individuals, that $\mathbf{p}_0$ does not depend on W_0^i, this is claimed by many researchers of asset pricing, which is true in this case. Let us write down the equilibrium price for any asset j

$$p_{0,j} = \frac{1}{\lambda_a}\left[y_j - \frac{1}{I}\Theta\sum_{k=1}^{N}\sigma_{jk}z_{m,k}\right]$$

(where σ_{jk} corresponds to the element in row j column k of Ω)

$$= \frac{1}{\lambda_a}\left[y_j - \frac{1}{I}\Theta Cov(\tilde{x}_j, \tilde{W}_m)\right] \tag{5.7}$$

By observing equation (5.6), the aggregate risk aversion coefficient Θ, should be negatively related with λ_a, ie. $\Theta \uparrow \Rightarrow \lambda_a \downarrow \Rightarrow \frac{1}{\lambda_a} \uparrow$. Therefore from this together with (5.7), we can deduce the following relationships between Θ and $p_{0,j}$,

Corollary 5.1. *If $\mathcal{B}_i = \mathcal{B} \quad \forall i$ and it remains constant, then*

$$\text{When } Cov(\tilde{x}_j, \tilde{W}_m) \geq 0, \quad \Theta \uparrow \quad \Rightarrow \quad \text{change in } p_{0,j} \text{ is uncertain.}$$
$$\text{When } Cov(\tilde{x}_j, \tilde{W}_m) < 0, \quad \Theta \uparrow \quad \Rightarrow \quad p_{0,j} \uparrow \tag{5.8}$$

for any asset j.

Corollary 5.1 has some interesting implications. When investors are homogeneous except they have different ARR, an increase in the aggregate ARR leads to lower returns for stocks whose payoffs are negatively correlated to that of the market portfolio. However, this becomes unclear when the payoffs are positively correlated.

Now we can compare the above relations with the case when there is a risk-free, in which $\lambda_a = R_f$, hence constant. Then Corollary 5.1 would become

$$\begin{aligned} &\text{When } Cov(\tilde{x}_j, \tilde{W}_m) \geq 0, && \Theta \uparrow && \Rightarrow && p_{0,j} \downarrow \\ &\text{When } Cov(\tilde{x}_j, \tilde{W}_m) = 0, && \Theta \uparrow && \Rightarrow && p_{0,j} \text{ is unchanged} \\ &\text{When } Cov(\tilde{x}_j, \tilde{W}_m) < 0, && \Theta \uparrow && \Rightarrow && p_{0,j} \uparrow \end{aligned} \tag{5.9}$$

which is a simpler relation, however, it does not hold anymore without the assumption a risk-free asset, because λ_a =zero-beta rate $+1$ is no longer a constant.

Since expected returns of any asset should be negatively related to its price and the fact that $\beta_j = Cov(\tilde{r}_j, \tilde{r}_m) = 1/(W_{m0}p_{0,j})Cov(\tilde{x}_j, \tilde{W}_m)$, we can re-write Corollary 5.1 in terms of expected return and beta

Corollary 5.2. *If $\mathcal{B}_i = \mathcal{B}$ for all i and it remains constant, then*

$$\begin{aligned} &\textit{When } \beta_j \geq 0, && \Theta \uparrow && \Rightarrow && \textit{change in } \mathbb{E}(\tilde{r}_j) \textit{ is uncertain.} \\ &\textit{When } \beta_j < 0, && \Theta \uparrow && \Rightarrow && \mathbb{E}(\tilde{r}_j) \downarrow \end{aligned} \tag{5.10}$$

for any asset j.

Furthermore, if there exists a risk-free asset with payoff $R_f > 0$, then we have

$$\begin{aligned} &\textit{When } \beta_j > 0, && \Theta \uparrow && \Rightarrow && \mathbb{E}(\tilde{r}_j) \uparrow \\ &\textit{When } \beta_j = 0, && \Theta \uparrow && \Rightarrow && \mathbb{E}(\tilde{r}_j) \textit{ is unchanged} \\ &\textit{When } \beta_j < 0, && \Theta \uparrow && \Rightarrow && \mathbb{E}(\tilde{r}_j) \downarrow \end{aligned} \tag{5.11}$$

for any asset j.

Corollary 5.2 is important in that it shows at least in the homogeneous case when the market belief stays constant that, increase in aggregate ARR will result in a drop in an asset's expected future return if the asset's beta is negative. This holds with or without the existence of a risk-free asset. When the beta coefficient is non-negative, with the assumption of a risk-free asset, the relation between expected return and the aggregate ARR is quite clear as set out in (5.15), however things becomes unclear when we relax the assumption of a risk-free asset.

Next, we want to see how does the change in risk-aversion of individual investors affect the aggregate risk aversion of the market and thus the equilibrium prices and expected returns. First, assume that every investor's risk aversion coefficient stays constant except for one, we can calculate the following partial derivative to see exactly what happens, for every θ_i,

$$\frac{\partial \Theta}{\partial \theta_i} = I \frac{\partial}{\partial \theta_i}\left[\left(\sum_{l=1}^{I} \theta_l^{-1}\right)^{-1}\right] = I\left(\sum_{l=1}^{I} \theta_l^{-1}\right)^{-2} \theta_i^{-2} = \frac{1}{I}\Theta^2 \theta_i^{-2} > 0. \tag{5.12}$$

Equation (5.12) shows that for any investor i, increase in his/her risk aversion coefficient will result in an increase in the aggregate risk aversion coefficient, the rate of increase is given by the second derivative,

$$\frac{\partial^2 \Theta}{\partial \theta_i^2} = \frac{2}{I}\Theta^2 \theta_i^{-4}\left(\frac{\Theta}{I} - \theta_i\right) \tag{5.13}$$

which would be negative if $1/\Theta < \theta$, this is likely to be the case if the number of investors (I) is large, so the rate of increase is higher for smaller θ_i and fewer investors. Equation (5.12) together with Corollary 5.1 and 5.2 will lead to the corresponding changes in the equilibrium prices and expected returns, which will be shown in the following corollary.

Corollary 5.3. *If $\mathcal{B}_i = \mathcal{B}$ for all i and the market remains constant except for a particular investor i, with ARR θ_i, who becomes more risk-averse, then*

$$\begin{aligned} &\text{When } \beta_j \geq 0, && \Rightarrow && \text{change in } p_{0,j} \text{ and } \mathbb{E}(\tilde{r}_j) \text{ are uncertain} \\ &\text{When } \beta_j < 0, && \Rightarrow && p_{0,j} \uparrow \quad \Rightarrow \mathbb{E}(\tilde{r}_j) \downarrow \end{aligned} \tag{5.14}$$

for any asset j.

Furthermore, if there exists a risk-free asset with payoff $R_f > 0$, then we have

$$\begin{aligned} &\text{When } \beta_j > 0, && \Rightarrow && p_{0,j} \downarrow \quad \Rightarrow \mathbb{E}(\tilde{r}_j) \uparrow \\ &\text{When } \beta_j = 0, && \Rightarrow && p0, j \text{ and } \mathbb{E}(\tilde{r}_j) \text{ are unchanged} \\ &\text{When } \beta_j < 0, && \Rightarrow && p_{0,j} \uparrow \quad \Rightarrow \mathbb{E}(\tilde{r}_j) \downarrow \end{aligned} \tag{5.15}$$

for any asset j.

The rate of increase/decrease is higher for smaller θ_i and fewer number of investors.

Lastly we consider a case when more than one investor changes their attitude towards risk, particularly, we consider a *Mean Preserved Spread*.

Definition 5.4. *Let $\{x_1, x_2, \cdots, x_n\}$ be a set of univariate observations, a MPS (Mean Preserved Spread) is a set of numbers $\{\epsilon_1, \epsilon_2, \cdots, \epsilon_n\}$ such that $\sum_{i=1}^n \epsilon_i = 0$, so if we let $x_i^{'} = x_i + \epsilon_i$, then $\bar{x}_i^{'} = \sum_{i=1}^n x_i + \epsilon_i = \sum_i^n x_i = \bar{x}$, thus the mean is preserved.*

In our case, we consider a MPS for the set of risk aversion coefficients, ie. $\{\theta_1, \theta_2, \cdots, \theta_I\}$, such that, let $\theta_i^{'} = \theta_i + \epsilon_i$,

$$1/I \sum_{i=1}^{I} (\theta_i^{'} - \bar{\theta}^{'})^2 \geq 1/I \sum_{i=1}^{I} (\theta_i - \bar{\theta})^2 \tag{5.16}$$

where $\bar{\theta}$ and $\bar{\theta}'$ denotes the mean of θ_i and θ_i', respectively, and $\bar{\theta} = \bar{\theta}'$. Condition (5.16) means that when investors' risk aversions change from θ_i to θ_i', there is a greater divergence of risk attitudes among the investors. We want to know whether this kind of MPS in the risk aversions will reduce the aggregate risk aversion, Chiarella, Deici and He (2006*a*) showed that this is true when $I = 2$, but for $I \geq 2$, we know this is in general not true from the case of $I = 2$.

5.3. **Homogeneous Variance/Covariance Matrices, Heterogeneous Expected payoffs.** In this case, we assume that investors agree on the variances and covariances of asset payoffs, but disagree on the expected future payoffs of the assets. So, $\Omega_i = \Omega$, which as we have seen in the previous subsection, implies that $\Omega_a = \Omega$. Thus, the equilibrium price vector is given by

$$\mathbf{p}_0 = \frac{1}{\lambda_a}\left[\mathbf{y}_a - \frac{1}{I}\Theta\Omega\mathbf{z}_m\right] \tag{5.17}$$

so for any asset j, its price is given by

$$p_{0,j} = \frac{1}{\lambda_a}\left[y_{a,j} - \frac{1}{I}\Theta Cov(\tilde{x}_j, \tilde{W}_m)\right] \tag{5.18}$$

where

$$\lambda_a = \frac{\mathbf{z}_m^T\mathbf{y}_a - \Theta\mathbf{z}_m^T\Omega\mathbf{z}_m/I}{W_{m0}} \qquad \text{and} \qquad y_{a,j} = \frac{1}{I}\sum_{i=1}^{I}\frac{\Theta}{\theta_i}y_{i,j} \tag{5.19}$$

From (5.18), it can be clearly seen that if there exists a risk-free asset, λ_a would be constant and an increase in an asset's aggregate expected payoff $y_{a,j}$ will increase its equilibrium price. However, without the assumption of a risk-free asset, since $\frac{1}{\lambda_a}$ is negatively related to $y_{a,j}$, the effect of $y_{a,j}$ on $p_{0,j}$ is uncertain, or at least more complex.

Now we are going look at the effect of changes in individual expected payoffs (ie. $\mathbf{y}_i$) on the aggregate expected payoffs. From (5.19), since risk aversion coefficients must be strictly positive, $y_{i,j}$ is positively related to $y_{a,j}$ for any asset j and any individual i given all other investors' expected payoffs of asset j remain constant, we can examine the exact relationship by calculating the following partial derivative,

$$\frac{\partial y_{a,j}}{\partial y_{i,j}} = \frac{1}{I}\frac{\Theta}{\theta_i} y_{i,j}. \tag{5.20}$$

Equation (5.20) indicates the positive relationship mentioned above, and the rate of increase in $y_{a,j}$ for an increase in $y_{i,j}$ is negatively related to investor i's risk aversion, which can be shown by taking second partial derivative w.r.t θ_i,

$$\frac{\partial^2 y_{a,j}}{\partial y_{i,j}\partial\theta_i} = \frac{y_{i,j}}{I}\left(\theta_i^{-1}\frac{\partial\Theta}{\partial\theta_i} - \theta_i^{-2}\Theta\right) = \Theta\theta_i^{-3} y_{i,j}/I(\Theta/I - \theta_i) \tag{5.21}$$

Obviously, the expression in (5.21) is likely to be negative if the number of investors (I) is large, which means a less risk averse investor's opinion would be more influential in the market. We should also add here that the absolute rate of increase would be very small if I is large. In other words, with homogeneous var/cov matrices, a particular individual's belief of an asset's expected payoff would have little effect on the asset's aggregate expected end-of-period payoff, regardless of his/her initial wealth. This is because the aggregate expected payoff vector ($\mathbf{y}_a$) is independent of individuals' shadow prices (λ_i^*), which is the only parameter affected by the initial wealth distribution.

Next, let us consider a MPS for the set of expected payoffs of assets, ie $\{y_{1,j}, y_{2,j}, \cdots, y_{I,j}\}$, the MPS is similar to the one in the previous subsection, which is a set of real numbers $\{\epsilon_{j,1}, \epsilon_{j,2}, \cdots, \epsilon_{j,n}\}$ such that $\sum_{i=1}^{n} \epsilon_{i,j} = 0$ and let $y'_{i,j} = y_{i,j} + \epsilon_{i,j}$,

$$1/I\sum_{i=1}^{I}(y'_{i,j} - \bar{y}'_j)^2 \geq 1/I\sum_{i=1}^{I}(y_{i,j} - \bar{y}_j)^2 \tag{5.22}$$

The condition in equation (5.22) means that when investors shift their belief of the expected payoff of asset j from $y_{i,j}$ to $y^{'}_{i,j}$, there is a greater divergence of opinions in terms of expected payoffs. According to the argument provided by Miller (1977) and the empirical tests performed in Diether, Malloy and Scherbina (2002), this kind of MPS in the expected payoffs should correspond to lower future returns, in other words $\mathbb{E}_a(\tilde{r}_j)$ would be reduced, the reason is that the over-optimistic investors will push up the price of the asset which then reduces its expected future returns. Let us see whether this is the case here in a simplified setting, when $I = 2$.

Example When $I = 2$, given $\epsilon > 0$ and consider 2 assets, namely asset j and k with$y_{2,j} < y_{1,j}$, and $y_{1,k} = y_{1,j} + \epsilon$ and $y_{2,k} = y_{2,j} - \epsilon$ [5], then

$$\begin{aligned} y_{a,j} &= (\theta_1^{-1} + \theta_2^{-1})^{-1}(\theta_1^{-1} y_{1,j} + \theta_2^{-1} y_{2,j}) \\ &= \frac{\theta_1^{-1}}{\theta_1^{-1} + \theta_2^{-1}} y_{1,j} + \frac{\theta_1^{-1}}{\theta_1^{-1} + \theta_2^{-1}} y_{2,j} \end{aligned} \tag{5.23}$$

and

$$y_{a,k} = \frac{\theta_1^{-1}}{\theta_1^{-1} + \theta_2^{-1}}(y_{1,j} + \epsilon)\frac{\theta_1^{-1}}{\theta_1^{-1} + \theta_2^{-1}}(y_{2,j} - \epsilon) \tag{5.24}$$

so

$$y_{a,j} - y_{a,k} = \frac{\epsilon}{\theta_1^{-1} + \theta_2^{-1}}(\theta_2^{-1} - \theta_1^{-1}) \tag{5.25}$$

According to (5.25), we have the following relations

$$\theta_1 < \theta_2 \qquad \Rightarrow \qquad y_{a,j} < y_{a,k} \tag{5.26}$$

$$\theta_2 < \theta_1 \qquad \Rightarrow \qquad y_{a,k} < y_{a,j} \tag{5.27}$$

(5.27) would be consistent with Miller's argument at least in the risk-free case, however, it will require investor 2 to be more risk averse than investor 1, so in other words,

[5] This specification already satisfies the condition in (5.22), which means the divergence of opinion about the asset's expected payoff is greater for asset k than for asset j

the investor with a more optimistic view of the asset's future payoff must be less risk averse. This suggests that belief of assets' expected future payoffs should be negatively related to risk aversion for any investor i.

5.4. **Homogeneous Expected payoffs, Heterogeneous Variance/Covariance Matrices.** Heterogeneity in the variance and covariance of assets' payoffs has always been hard to deal with. The equilibrium price for any asset j was shown in equation (5.18), from which it can be seen that if there exists a risk-free asset ($\lambda_a = R_f$), then higher the aggregate covariance between asset j's future payoff and the market payoff, lower the equilibrium price of asset j, hence higher the aggregate expected return (ie. $Cov_a(\tilde{x}_j, \tilde{W}_m) \uparrow \Rightarrow p_{0,j} \downarrow \Rightarrow \mathbb{E}_a(\tilde{r}_j) \uparrow$), however, this relation does not hold anymore without the assumption of a risk-free asset since the aggregate shadow price is negatively related to $Cov_a(\tilde{x}_j, \tilde{W}_m)$, hence $Cov_a(\tilde{x}_j, \tilde{W}_m) \uparrow \Rightarrow 1/\lambda_a \uparrow$ and the change in $p_{0,j}$ is uncertain. Also, mathematically, the exact relation between $Cov_i(\tilde{x}_j, \tilde{W}_m)$ and $Cov_a(\tilde{x}_j, \tilde{W}_m)$ is uncertain, although intuitively, one would expect a positive relationship. It is also difficult to consider the effect of a MPS in the variances/covariances matrices although Chiarella, Deici and He (2006) showed that in the case when $I = 2$ with the assumption of a risky asset and the risky assets are uncorrelated, if greater divergence of opinion in the variance of asset j is to reduce its aggregate variance, then we need the investor who believes a lower variance for asset j (less risky) to be less risk averse.[6] This is consistent with our discussion of MPS in the expected payoffs and also the argument in Miller (1977). The intuition we obtain from the above discussion and also the discussion in the last subsection is that an optimistic investor should be less risk aversion compare to a pessimistic investor. If we put that in mathematical

[6]Since assets returns are uncorrelated, $Cov_a(\tilde{x}_j, \tilde{W}_m) = Var_a(\tilde{x}_j)$, then if we assume homogeneous belief in expected payoff of assets, this will imply a rise in equilibrium price of asset j and a decrease in its expected future returns

terms, it means

$$\theta_i \leq \theta_l \qquad \Leftrightarrow \qquad Cov_i(\tilde{x}_j, \tilde{W}_m) \leq Cov_l(\tilde{x}_j, \tilde{W}_m) \qquad \text{and} \qquad y_{l,j} \leq y_{i,j} \quad (5.28)$$

Unfortunately, the relation in (5.28) is not yet built into our model. Chiarella, Deici and He (2006) also proposed that

$$\mathbf{x}^T \Omega_a \mathbf{x} \leq \mathbf{x}^T \bar{\Omega} \mathbf{x} \qquad \text{for any} \qquad \mathbf{x} \quad (5.29)$$

where

$$\bar{\Omega} = \frac{1}{I} \sum_{i=1}^{I} \frac{\Theta}{\theta_i} \Omega_i$$

and proved it with the assumption of risk-free asset and asset returns are uncorrelated. If (5.29) is true in general, it will suggest that the market's belief of variance for any portfolio is always less than or equal to the average belief (weighted by risk aversion coefficients) of variance in the market when there is heterogeneity in the variance/covariance matrices. This is important, because it explains why so many empirical tests of the CAPM has failed and the reason is that they failed to account for the heterogeneity in the market. This paper provides a proof for (5.29) in the case when $I = 2$ with the assumption of a risk-free asset and it is in Appendix A.

6. Implications on the portfolio frontier

In this section, we will discuss the implications of our model on the market portfolio frontier. We know that in market equilibrium, every investor will hold his/her equilibrium optimal portfolio expressed in equation (3.8) and the market portfolio is obviously $\mathbf{z}_m$. Now the question is that with heterogeneous beliefs amongst the investors, are these optimal portfolios and the market portfolio still on the efficient part of the frontier (or mean-variance efficient) as they were in the homogenous (traditional case)? We will discuss this question first in the general case where we do not make the assumption of a risk-free asset.

If we assume now that the consensus belief $\mathcal{B}_a = \{\Omega_a, \mathbf{y}_a\}$ is the markets belief, ie. $\mathcal{B}_M = \mathcal{B}_a$. Now consider a consensus investor in our market with market beliefs $\mathcal{B}_a$, risk aversion coefficient θ_i and initial wealth W_0^i, his/her equilibrium optimal portfolio will be given by according to equation (3.8)

$$\mathbf{z}_i^* = \left(1 - \frac{\lambda_i^*}{\lambda_a}\right)\theta_i^{-1}\Omega_a^{-1}\mathbf{y}_a + \frac{1}{I}\theta_i^{-1}\Theta\frac{\lambda_i^*}{\lambda_a}\mathbf{z}_m \tag{6.1}$$

(6.1) shows that any consensus investor will divide his/her investment into two portfolios, namely, $\Omega_a^{-1}\mathbf{y}_a$ and the market portfolio $\mathbf{z}_m$, which is consistent with the *Two Fund Separation Theorem*[7] and such portfolios must be mean-variance efficient due to the utility function of our investors, hence they must be on the efficient part of the frontier, which means that the portfolios $\Omega_a^{-1}\mathbf{y}_a$ and $\mathbf{z}_m$ must be also frontier portfolios. If we take the sum of both side of equation (6.1), we will obtain

$$\sum_{i=1}^{I}\mathbf{z}_i^* = \sum_{i=1}^{I}\left(1 - \frac{\lambda_i^*}{\lambda_a}\right)\theta_i^{-1}\Omega_a^{-1}\mathbf{y}_a + \mathbf{z}_m \tag{6.2}$$

(6.2) with the market aggregation condition (3.1)suggests that the net supply of the portfolio $\Omega_a\mathbf{y}_a$ is zero, ie.

$$\sum_{i=1}^{I}\left(1 - \frac{\lambda_i^*}{\lambda_a}\right)\theta_i^{-1}\Omega_a^{-1}\mathbf{y}_a = \mathbf{0}$$

and the net supply of the market portfolio is of course 1, therefore the market portfolio ($\mathbf{z}_m$) must be on the efficient part of the frontier since the market portfolio is a convex combination of individual consensus investors' optimal portfolios, which are mean-variance efficient. The above arguments also hold when there exists a risk-free asset in

[7]See Huang and Lizenberger (1988) Chapter 4, page 83

the market. However, when an investor subjective belief ($\mathcal{B}_i$) differs from the market belief ($\mathcal{B}_a$), it is not clear, at least analytically, whether their equilibrium optimal portfolio lies on the efficient frontier. Because of that, we provide the following numerical example, in which we have 2 investors and 3 risky assets, individuals' risk aversion coefficients, subjective beliefs about the market and initial wealth are all predetermined, we calculate the equilibrium price vector, form the consensus belief and calculate the aggregate expected returns and variance/covariance of asset returns, finally construct the portfolio frontier and locate the market portfolio and individual's optimal portfolio. We do the above in both case, in which we have and have not a risk-free asset and observe the change in the frontier.

Example We assume two investors in the market ($I = 2$) and three risky assets ($N = 3$).

Initial Wealth	Shadow Price	Risk Aversion	Expected payoffs	Variance/Covariance of payoffs
$W_0^1 = 10$	$\lambda_1^* = 0.7894$	$\theta_1 = 5$	$\mathbf{y}_1 = \begin{pmatrix} 6.60 \\ 9.35 \\ 9.78 \end{pmatrix}$	$\Omega_1 = \begin{pmatrix} 0.6292 & 0.1553 & 0.2262 \\ & 0.7692 & 0.1492 \\ & & 2.1381 \end{pmatrix}$
$W_0^2 = 10$	$\lambda_2^* = 1.6520$	$\theta_2 = 1$	$\mathbf{y}_2 = \begin{pmatrix} 9.60 \\ 12.35 \\ 12.78 \end{pmatrix}$	$\Omega_2 = \begin{pmatrix} 0.4292 & -0.0447 & 0.0262 \\ & 0.5692 & -0.0508 \\ & & 1.7381 \end{pmatrix}$

TABLE 6.1. Initial market specifications and individuals' subjective beliefs

First, let us consider the case in which we do not have a risk-free asset. The equilibrium price vector can be calculated using the information in table-6.1 by equation

(3.2) [8], assuming there is one share available for each asset, ie. $\mathbf{z}_m = \begin{pmatrix} 1 & 1 & 1 \end{pmatrix}^T$

$$\mathbf{p}_0 = \begin{pmatrix} 5.6436 & 7.4328 & 6.9236 \end{pmatrix}^T$$

with each investor's optimal demand given by

$$\mathbf{z}_1^* = \begin{pmatrix} 0.380 & 0.768 & 0.310 \end{pmatrix}^T \qquad \mathbf{z}_2^* = \begin{pmatrix} 0.620 & 0.232 & 0.690 \end{pmatrix}^T$$

Next we will construct the consensus belief $\mathcal{B}_a$ as well as the aggregate risk aversion coefficient Θ, the aggregate shadow price λ_a by using Proposition 3.2. It will lead to the following result,

Total Market Initial Wealth	Shadow Price	Risk Aversion	Expected payoffs	Variance/Covariance of payoffs
$W_{m0} = 20$	$\lambda_a = 1.5083$	$\Theta = 1.6667$	$\mathbf{y}_a = \begin{pmatrix} 8.88 \\ 11.63 \\ 12.06 \end{pmatrix}$	$\Omega_a = \begin{pmatrix} 0.4383 & -0.0356 & 0.0352 \\ & 0.5783 & -0.0417 \\ & & 1.9472 \end{pmatrix}$

TABLE 6.2. Aggregate market beliefs, shadow price and ARR

Now to calculate the variance/covariance matrices, expected payoffs for each investor and the market, we need to do the following, let

$$P_0 = diag[\mathbf{p}_0] = \begin{pmatrix} 5.6436 & 0 & 0 \\ & 7.4328 & 0 \\ & & 6.9236 \end{pmatrix}$$

[8]Excel Solver is used to solve for the price vector implicitly, to the accuracy of 10^{-6}

and

$$\mathbb{E}_i(\mathbf{r}) := P_0^{-1}\mathbf{y}_i - \mathbf{1} \qquad V_i := P_0^{-1}\Omega_i P_0^{-1} \qquad i = 1, 2, a$$
$$\mathbf{w}_i^* := \frac{1}{W_0^i} P_0 \mathbf{z}_i^* \qquad \mathbb{E}_i(r_{ip}^*) := \mathbb{E}_i(\mathbf{r})^T \mathbf{w}_i^* \qquad \sigma_{ip}^* = (\mathbf{w}_i^{*T} V_i \mathbf{w}_i^*)^{1/2} \qquad i = 1, 2$$
$$\mathbb{E}_a(r_{ip}^*) := \mathbb{E}_a(\mathbf{r})^T \mathbf{w}_i^* \qquad \sigma_{ip}^a = (\mathbf{w}_i^{*T} V_a \mathbf{w}_i^*)^{1/2} \qquad i = 1, 2$$
$$\mathbf{w}_m := \frac{1}{W_m} P_0 \mathbf{z}_m \qquad \mathbb{E}_a(r_m) := \mathbb{E}_a(\mathbf{r})^T \mathbf{w}_m \qquad \sigma_{am} = (\mathbf{w}_m^T V_a \mathbf{w}_m)^{1/2} \qquad \beta := V_a \mathbf{w}_m \tag{6.3}$$

In the definitions made in (6.3), $\mathbb{E}_i(\mathbf{r})$ and V_is are the expected returns vectors and var/cov matrices in terms of asset returns for each investor and the aggregate market, then subsequently, $\mathbf{w}_i^*$ are the individuals' optimal portfolio weights, $\mathbb{E}_i(r_{ip}^*)$ and σ_{ip}^* are the individuals' expected portfolio return and standard deviations, respectively, under their subjective beliefs, $\mathcal{B}_i$, $\mathbb{E}_a(r_{ip}^*)$ and σ_{ip}^a take similar meanings under the aggregate market's belief, $\mathcal{B}_a$. $\mathbf{w}_m$ is the market portfolio weights vector, $\mathbb{E}_a(r_m)$ and σ_{am} are the market return and volatility under the aggregate belief respectively, finally β is the vector of beta coefficients. Now according to these definitions, we can calculate their numerical values in our case.

Expected returns	Variance/Covariance of returns	optimal portfolio weights	Portfolio Return/SD	
$\mathbb{E}_1(\tilde{\mathbf{r}}) = \begin{pmatrix} .1690 \\ .2577 \\ .4126 \end{pmatrix}$	$V_1 = \begin{pmatrix} .0198 & .0037 & .0058 \\ & .0139 & .0029 \\ & & .0446 \end{pmatrix}$	$\mathbf{w}_1^* = \begin{pmatrix} .2144 \\ .5711 \\ .2145 \end{pmatrix}$	$\mathbb{E}_1(r_{1p}^*) = .2719$ $\sigma_{1p}^* = .09824$	$\mathbb{E}_a(r_{1p}^*) = .6043$ $\sigma_{1p}^a = .0748$
$\mathbb{E}_2(\tilde{\mathbf{r}}) = \begin{pmatrix} .7006 \\ .6613 \\ .8459 \end{pmatrix}$	$V_2 = \begin{pmatrix} .0135 & -.0011 & .0007 \\ & .0103 & -.0010 \\ & & .0404 \end{pmatrix}$	$\mathbf{w}_2^* = \begin{pmatrix} .3499 \\ .1722 \\ .4778 \end{pmatrix}$	$\mathbb{E}_2(r_{2p}^*) = .7633$ $\sigma_{2p}^* = .1054$	$\mathbb{E}_a(r_{2p}^*) = .6522$ $\sigma_{2p}^a = .1065$
$\mathbb{E}_a(\tilde{\mathbf{r}}) = \begin{pmatrix} .5729 \\ .5644 \\ .7418 \end{pmatrix}$	$V_a = \begin{pmatrix} .0138 & -.0008 & .0009 \\ & .0105 & -.0008 \\ & & .0406 \end{pmatrix}$	$\mathbf{w}_m = \begin{pmatrix} .2822 \\ .3716 \\ .3462 \end{pmatrix}$		$\mathbb{E}_a(r_m) = .6283$ $\sigma_{a,m} = .0848$

$$\boldsymbol{\beta} = \begin{pmatrix} 0.5390 & 0.4681 & 1.9468 \end{pmatrix}^T$$

TABLE 6.3. Individual and market's status at market equilibrium

Now with the information provided in Table-6.3, we can now construct the portfolio frontiers for each investor and for the aggregate market in the mean-standard deviation space, and locate the optimal portfolio for individual investors as well as the market portfolio. Construction of the portfolio is based on the idea of minimizing the variance for a given expected return, for more detail, refer to section 3 of Huang and Litzenberger (1988). Figure-6.1 exhibit such a graph.

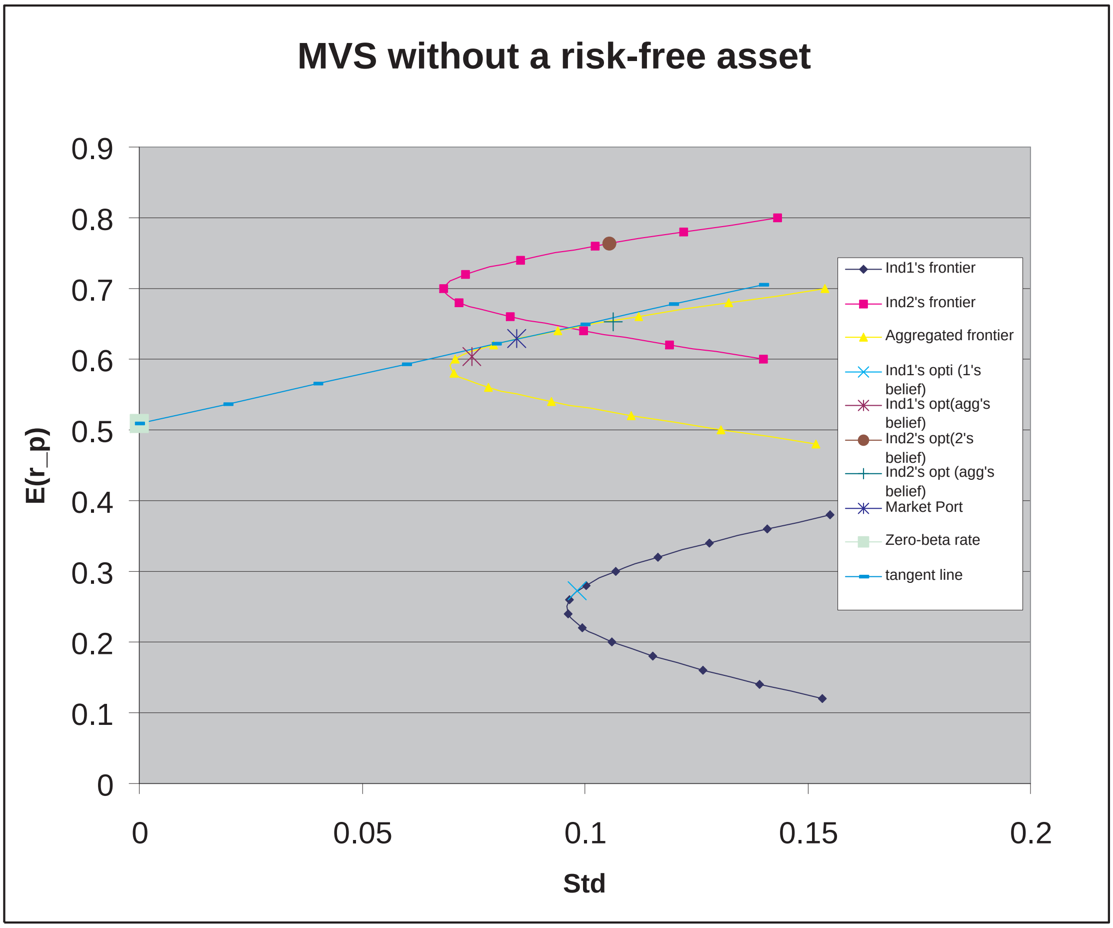

FIGURE 6.1. Individual and the aggregate portfolio frontiers without a risk-free asset

Figure-6.1 shows that without a risk-free asset, the aggregate market's portfolio frontier is between two individuals' frontier, however, it is closer to that of investor

2, the reason is that investor 2 is less risk averse and also more optimistic about the market, hence he/she is a stronger driving force of the market. It can be seen that under the belief of investor 1, his/her optimal portfolio is on the portfolio frontier under his/her belief and the same holds for investor 2, this is no surprise since the nature of their utility function is to find the most desirable mean-variance efficient portfolio according to their degree of risk-aversion measured by ARR. However, under the aggregate market belief, although the market portfolio looks like it is safely on the efficient frontier, whether individual investors' optimal portfolios are on the frontier is questionable. Because Figure-6.1 is not clear enough to show the exact location of the individual optimal portfolios, we provide a zoom-in version to observe things clearly near the efficient part of the aggregate market's portfolio frontier, Figure-6.2 is such a graph. We should also mention here that the tangent relation still hold in this case, ie. the tangent line of the aggregate frontier at the point of the market portfolio has a y-intercept equal to the zero-beta rate, which is also true in the homogenous case.

Figure-6.2 clearly shows that under the aggregate market's belief, while the market portfolio is on the efficient frontier, both investors' optimal portfolios are under the efficient frontier, thus not mean-variance efficient under aggregate belief. If one thinks carefully, this result is also intuitively correct, since both investors made ”wrong guesses” about the market, investor 1 being too pessimistic and investor 2 being over-optimistic, their optimal portfolios suffer from those ”wrong guesses” in terms of mean-variance efficiency. But this does not affect the position of the market portfolio as it is still safely on the efficient frontier, because being the most diversified portfolio in the market, any effects coming from the ”wrong guesses” will be averaged out to make it the ”correct guess”. Platen (2006) showed through the *Diversification Theorem* that any diversified portfolio in the market is an approximate *Growth Optimal*

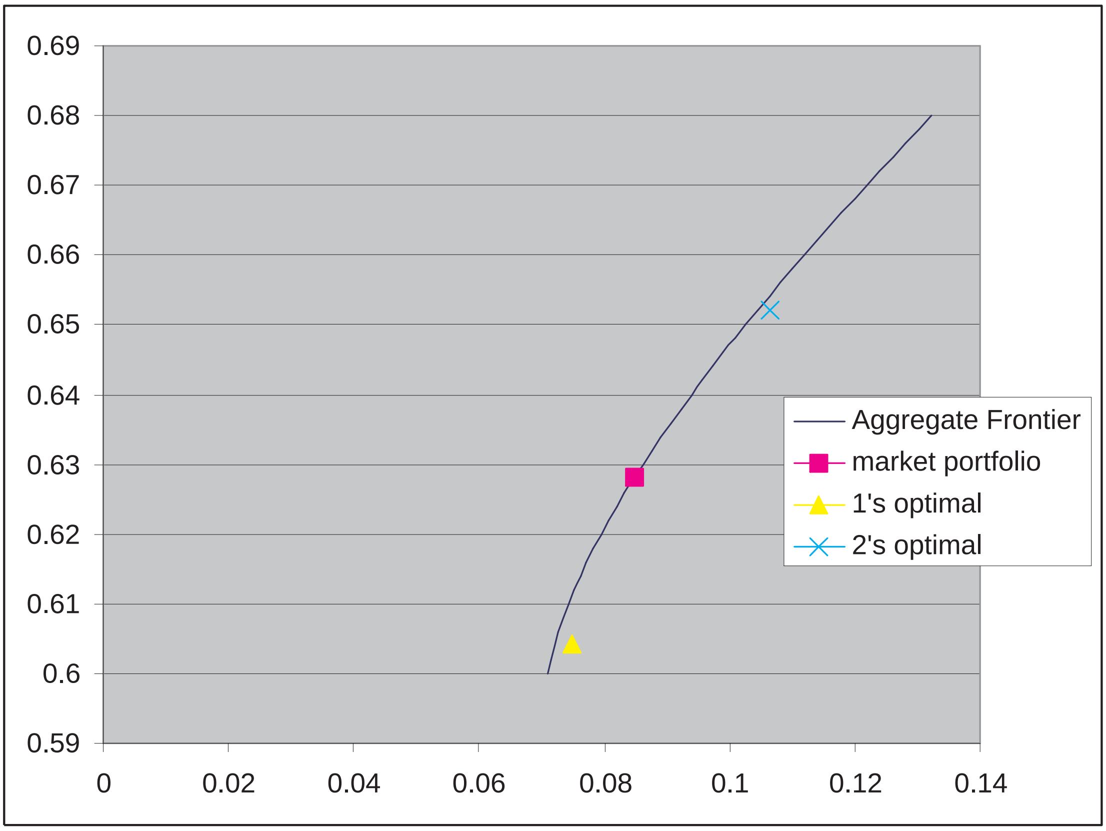

FIGURE 6.2. Close-up of the locations of individuals' optimal portfolios and the market portfolio

Portfolio (GOP), which is the best performing portfolio that can not be beaten systematically. Since the market portfolio is certainly diversified in any reasonable case, so this paper supports the claim in Platen (2006) that a diversified portfolio such as the market portfolio is the best performing portfolio in the market. Sharpe (2007) also found through simulations of market trading by his program APSIM that at market equilibrium, the market portfolio performs nearly as good in terms of sharpe ratio as an investor who makes the "correct guess" about the probability distribution of the future states of the market. Although he assumed that a risk-free asset exists, his findings are still consistent with ours, just serves as a special case.

Moreover, we can now add a risk-free asset (f) in our market with current price of 1 and future payoff $R_f = 1.05$, as mentioned before, this implies that all shadow price are now equal to R_f, ie. $\lambda_1^* = \lambda_2^* = \lambda_a = R_f$, other characteristics of the market will adjust accordingly. First, we can re-calculate the values in Table-6.2,

Total Market Initial Wealth	Shadow Price	Risk Aversion	Expected payoffs	Variance/Covariance of payoffs
$W_{m0} = 20$	$\lambda_a = 1.05$	$\Theta = 1.6667$	$\mathbf{y}_a = \begin{pmatrix} 9.3257 \\ 12.0767 \\ 12.5084 \end{pmatrix}$	$\Omega_a = \begin{pmatrix} 0.4474 & -0.0266 & 0.0443 \\ & 0.5873 & -0.0327 \\ & & 1.9562 \end{pmatrix}$

TABLE 6.4. Aggregate market beliefs, shadow price and ARR with a risk-free asset

from which the equilibrium price vector is found to be

$$\mathbf{p}_0 = \begin{pmatrix} 8.5125 & 11.0825 & 10.3511 \end{pmatrix}$$

with each investor's optimal demand given by

$$\mathbf{z}_1^* = \begin{pmatrix} -0.628 & -0.468 & -0.003 \end{pmatrix}^T \qquad z_{1,f} = 20.554$$

$$\mathbf{z}_2^* = \begin{pmatrix} 1.628 & 1.468 & 1.003 \end{pmatrix}^T \qquad z_{2,f} = -30.500$$

The equilibrium prices are higher than the case without a risk-free asset. This is because in this example, the risk-free rate is much lower than the original zero-beta rate, which reduced the aggregate shadow price, and thus increased the equilibrium asset prices. Now with the equilibrium price vector found, we can re-calculate the values in Table-6.3 and construct the individual frontiers as well as the aggregate market frontier and locate individual optimal portfolio as well as the market portfolio. We should also mention that now with a risk-free asset, investor 1 is short-selling risky assets to invest in the risk-free asset while investor 2 is borrowing at the risk-free rate to invest in the risky assets,a very different situation to before. Also the risk-free asset is not in net zero supply as assumed in the classical economic literatures, it is in strictly negative supply, if we were like to have its net supply zero, then the risk-free rate has to be increased.

Expected returns	Variance/Covariance of returns	optimal portfolio weights	Portfolio Return/SD	
$\mathbb{E}_1(\tilde{\mathbf{r}}) = \begin{pmatrix} -.2250 \\ -.1565 \\ -.0552 \end{pmatrix}$	$V_1 = \begin{pmatrix} .0087 & .0016 & .0026 \\ & .0063 & .0013 \\ & & .0200 \end{pmatrix}$	$\mathbf{w}_1^* = \begin{pmatrix} -.5342 \\ -.5183 \\ -.0029 \end{pmatrix}$ $w_{1,f} = 2.0554$	$\mathbb{E}_1(r_{1p}^*) = .3042$ $\sigma_{1p}^* = .0713$	$\mathbb{E}_a(r_{1p}^*) = .0046$ $\sigma_{1p}^a = .0538$
$\mathbb{E}_2(\tilde{\mathbf{r}}) = \begin{pmatrix} .1274 \\ .1142 \\ .2347 \end{pmatrix}$	$V_2 = \begin{pmatrix} .0059 & -.0005 & .0003 \\ & .0046 & -.0004 \\ & & .0181 \end{pmatrix}$	$\mathbf{w}_2^* = \begin{pmatrix} 1.3854 \\ 1.6266 \\ 1.0380 \end{pmatrix}$ $w_{2,f} = -3.0500$	$\mathbb{E}_2(r_{2p}^*) = .4535$ $\sigma_{2p}^* = .2009$	$\mathbb{E}_a(r_{2p}^*) = .3421$ $\sigma_{2p}^a = .2083$
$\mathbb{E}_a(\tilde{\mathbf{r}}) = \begin{pmatrix} .0955 \\ .0897 \\ .2084 \end{pmatrix}$	$V_a = \begin{pmatrix} .0062 & -.0003 & .0005 \\ & .0048 & -.0003 \\ & & .0183 \end{pmatrix}$	$\mathbf{w}_m = \begin{pmatrix} .4256 \\ .5541 \\ .5176 \end{pmatrix}$ $w_{m,f} = -.4973$		$\mathbb{E}_a(r_m) = .1734$ $\sigma_{a,m} = .0860$

$$\boldsymbol{\beta} = \begin{pmatrix} 0.3690 & 0.3218 & 1.2841 \end{pmatrix}^T$$

TABLE 6.5. Individual and market's status at market equilibrium with a risk-free asset

From Figure-6.3, now with a risk-free asset, the relationships between risk and return becomes linear, frontier under investor 1's belief has the highest slope follow by investor 2 and the aggregate market's frontier has the smallest slope. So, investor 1 who was pessimistic previously is now the more optimistic investor. This is because that the slope of the efficient frontiers is given by

$$\frac{\mathbb{E}_i(\tilde{r}_p) - r_f}{\sigma_i(\tilde{r}_p)} = \sqrt{H_i}, \qquad H_i = (\mathbb{E}_i(\tilde{\mathbf{r}}) - r_f\mathbf{1})^T V_i^{-1}(\mathbb{E}_i(\tilde{\mathbf{r}}) - r_f\mathbf{1}), \qquad i = 1, 2, a \tag{6.4}$$

[9]

[9] Again, for the details of the derivation of this expression, refer to Huang and Litzenberger (1988)

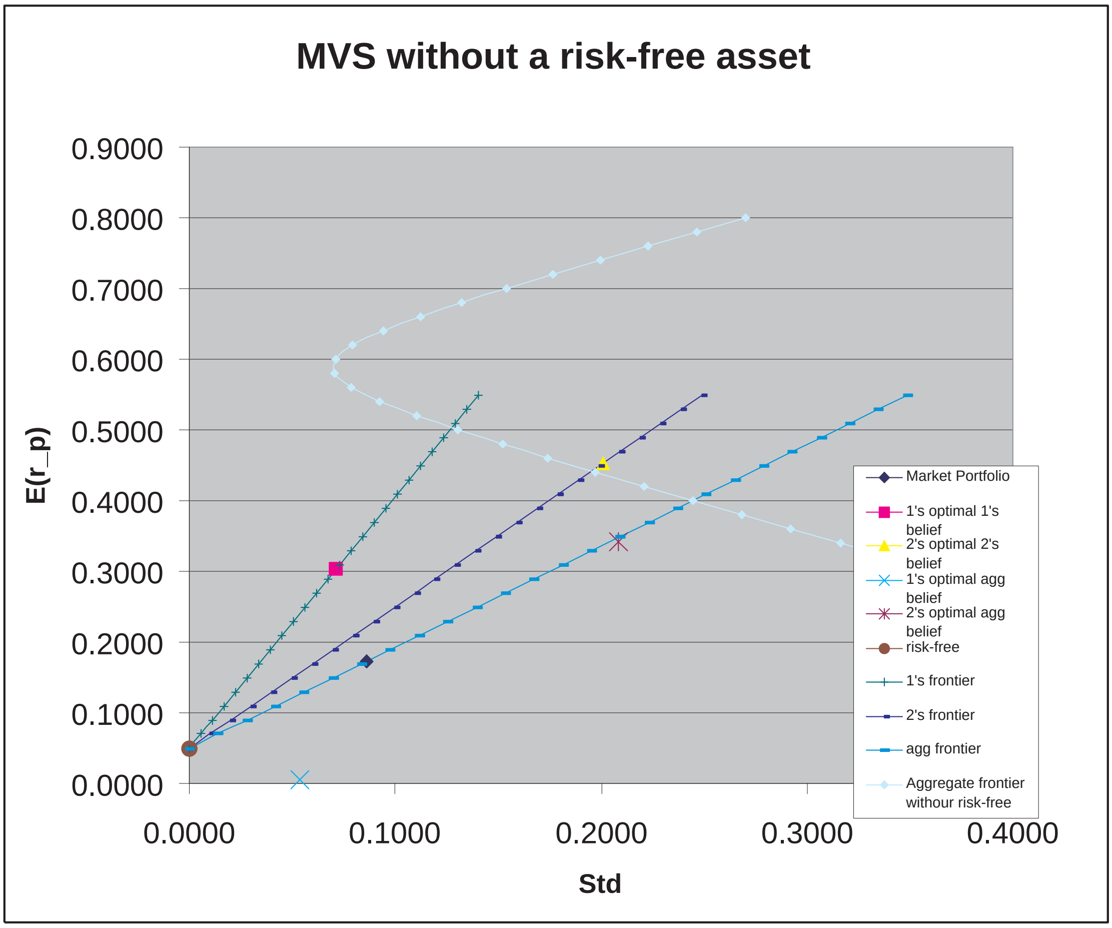

FIGURE 6.3. Individual and the aggregate portfolio frontiers without a risk-free asset

From equation (6.4), it is clear that H_i would be larger if V_i is smaller [10], and larger $|\mathbb{E}_i(\tilde{r}_j) - r_f|$. Hence, the slope of the frontier is higher for an investor who believes that the asset returns have a small variance and covariance and expected returns are very different to the risk-free rate. In our example, although investor 1's variance/covaraince matrix is slightly larger than investor 2's, but this is not enough to offset the larger difference between his/her belief of expected returns of the risky asset and the risk-free

[10]Here the way we compare the varirance/covaraince matrices is that, $V_i \leq V_j \leftrightarrow \mathbf{x}^T V_i \mathbf{x} \leq \mathbf{x}^T V_j \mathbf{x}$ for any vector $\mathbf{x}$, in other words, $V_j - V_i$ is a semi-positive definite matrix.

rate, thus investor 1's frontier has the highest slope. On the other hand, market aggregate frontier has the smallest slope, because its belief of expected return of risky assets is closest to the risk-free rate. Therefore, we can see from the discussion above that when there exists a risk-free asset in the market, the slopes of the individual and aggregate market's frontiers depend on the level of the risk-free return, and it is certainly not true that a optimistic investor's frontier definitely has a higher slope than a pessimistic investor's frontier, also the aggregate market's frontier does not necessarily locate between individual frontiers. We can also see that in this case, investor 1's optimal portfolio under the aggregate market's belief is well below the aggregate market's efficient frontier compare with investor 2 optimal portfolio, this is because investor 1's frontier deviates much more from the aggregate market's frontier than investor 2, so his/her optimal portfolio suffers in terms of efficiency as a result.

Lastly, what we can say from observing Figure-6.3 is that the tangency relation between the portfolio frontier without a risk-free asset and the frontier with a risk-free asset no longer hold, the reason is that when a risk-free asset is added to the market, shadow prices for individual investors and the aggregate market have all changed to the risk-free asset's payoff, as a result, the variance/covariance matrix of asset returns and the expected returns vector does not remain the same under the aggregated belief as we introduce the risk-free asset into the market, thus there is no tangency relation between the two frontiers. We would also like to mention here that because we assume in our model that investors form their beliefs of the market in terms of expected asset payoffs and variance/covaraince of asset payoffs, so individuals' and aggregate market's implied beliefs in terms of asset returns depend on also the equilibrium asset prices, therefore even in the homogeneous case, aggregated beliefs in terms of asset return does not necessarily stay the same when a risk-free asset is added, because the equilibrium asset prices have changed. This is inconsistent with the traditional finance theory and to overcome this problem, one can allow individual investors to form their beliefs in terms of asset returns and variance/covaraince of asset returns, then in the homogeneous case, one can recover the tangency relation, which we will talk about in the next section.

7. An Alternative Set up

In this section, we will try to recover the tangency relation between the frontier with a risk-free asset and the frontier without in the homogeneous case. This is important since this is a standard feature of the Markowitz efficient frontiers in the traditional finance theory. We will do this by allowing investors in our market to form their beliefs of the market in terms of expected asset returns and variance/covaraince of asset returns. To be consistent with section 6, we let $\mathbb{E}_i(\tilde{\mathbf{r}})$ be the expected asset returns vector and V_i be the variance/covariance matrix of asset returns for investor i. Denote $\mathcal{B}_i = (\mu_i, V_i)$ as the set of subjective beliefs of investor i, and let

$$\boldsymbol{\pi}_i = (\pi_{i,1}, \pi_{i,2} \cdots, \pi_{i,I})^T$$

be the investment of investor i in the N assets in dollar amount, and $W_{i,0}$ be the initial wealth of investor i in dollar amount.

7.1. **Individual's Portfolio Selection.** Under assumption (H1) and (H2), a particular investor i's end-of-period maximization problem becomes,

$$\max_{\boldsymbol{\pi}_i} \left[\boldsymbol{\mu}_i^T \boldsymbol{\pi}_i - \frac{\theta_i}{2} \boldsymbol{\pi}_i V_i \boldsymbol{\pi}_i \right]$$

subject to the wealth constraint

$$\boldsymbol{\pi}_i^T \mathbf{1} = W_{i,0}. \tag{7.1}$$

where $\boldsymbol{\pi}_i = \mathbb{E}_i(\tilde{\mathbf{r}}_i)+\mathbf{1}$. We can solve this optimization problem and yield the following result,

$$\boldsymbol{\pi}_i^* = \theta_i^{-1}V_i^{-1}(\boldsymbol{\mu}_i - \lambda_i^*\mathbf{1}) \tag{7.2}$$

where

$$\lambda_i^* = \frac{\mathbf{1}^T V_i^{-1}\boldsymbol{\mu}_i - \theta_i W_{i,0}}{\mathbf{1}^T V_i^{-1}\mathbf{1}} \tag{7.3}$$

[11] Clearly, λ_i^* is still the shadow price of investor i, however it takes a slightly different meaning here, we have

$$\lambda_i^* = \frac{\partial Q_i(\boldsymbol{\pi}^*)}{\partial \pi_{i,j}} \qquad \forall j$$

where

$$Q_i(\boldsymbol{\pi}) := \boldsymbol{\mu}_i^T\boldsymbol{\pi}_i - \frac{\theta_i}{2}\boldsymbol{\pi}_i^T V_i \boldsymbol{\pi}_i$$

is the certainty equivalent end-of-period wealth of investor i in terms of his dollar investment in each asset, so λ_i^* is the *optimal marginal certainty equivalent end-of-period wealth per dollar investment in asset j for investor i* and it is constant for all j. Obviously, if there exist a risk-free asset (f) with return r_f, then it must be true that $\lambda_i^* = 1 + r_f = \mu_f$ for any investor i.

7.2. **Market Aggregation and Consensus Belief.** So far, everything appears to be consistent with the previous set up, now let us see whether we can still construct a belief in our new defined market, under our new aggregate condition

$$\boldsymbol{\pi}_m = \sum_{i=1}^{I} \boldsymbol{\pi}_i^*. \tag{7.4}$$

The equilibrium price vector generated in terms of each investor's subjective beliefs is

$$\mathbf{p}_0 = Z^{-1}\sum_{i=1}^{I}\theta_i^{-1}V_i^{-1}(\boldsymbol{\mu}_i - \lambda_i^*\mathbf{1}) \tag{7.5}$$

[11]Proofs in this section are omitted since they are all quite similar to that of section 2, 3 and 4, there is no need to repeat them here

where Z is a $N\times N$ diagonal matrix with diagonal elements z_{jj} representing the market supply in absolute amount of asset j. It turns out that we can still find a consensus belief such that, if held by all investors will generate the same equilibrium price vector as in equation (7.5).

Proposition 7.1. *Under assumptions (H1) and (H2), let*

$$\Theta := \left(\frac{1}{I}\sum_{i=1}^{I}\theta_i^{-1}\right)^{-1}, \tag{7.6}$$

$$\lambda_a := \frac{1}{I}\Theta\sum_{i=1}^{I}\theta_i^{-1}\lambda_i^*. \tag{7.7}$$

Then

(i) *the unique consensus belief $\mathcal{B}_a$ is given by*

$$V_a = \Theta^{-1}\lambda_a\left(\frac{1}{I}\sum_{i=1}^{I}\lambda_i^*\theta_i^{-1}V_i^{-1}\right)^{-1}, \tag{7.8}$$

$$\boldsymbol{\mu}_a = \mathbb{E}_a(\mathbf{1}+\tilde{\mathbf{r}}) = \Theta V_a\left(\frac{1}{I}\sum_{i=1}^{I}\theta_i^{-1}V_i^{-1}\boldsymbol{\mu}_i\right); \tag{7.9}$$

[12]

(ii) *the market equilibrium price* $\mathbf{p}_o$ *is determined by*

$$\mathbf{p}_0 = I\Theta Z^{-1}V_a^{-1}(\boldsymbol{\mu}_a - \lambda_a\mathbf{1}); \tag{7.10}$$

Therefore, we see from Proposition 7.1 that both equilibrium and a consensus belief still exist even if we allow investors to form their beliefs of the market in terms of asset return instead of asset payoffs. Also the new set up has some advantages, (1) since the shadow price λ_i^* no longer depends on the equilibrium prices, the equilibrium price vector can now be solved explicitly, which is a computational convenience, (2) as

[12]This expression can be easily re-written as $\mathbb{E}_a(\tilde{\mathbf{r}}) = \Theta V_a\left(\frac{1}{I}\sum_{i=1}^{I}\theta_i^{-1}V_i^{-1}\mathbb{E}_i(\tilde{\mathbf{r}})\right)$

we mentioned before, the new set up allow us to recover the tangency relation in the standard homogeneous case, the reason that we can do this now but not before is that with the new set up, individuals' beliefs in terms of asset return does not vary with changes in equilibrium prices, so that in the homogeneous case, market belief will not change even though the equilibrium prices may alter when a risk-free asset is added to the market resulting in all shadow prices converging to one plus the risk-free rate.

However, in the heterogenous case, we will only recover the tangency relation if the belief of variance/covaraince of asset returns are homogeneous, ie. $V_i = V$, which implies that $V_a = V$ from equation (7.8), because then if a risk-free asset is added to the market, V_a will not be affected, thus $\mathbb{E}_a(\tilde{\mathbf{r}})$ also remains the same, hence the portfolio frontiers will exhibit the tangency relation since the market aggregate belief in terms of asset returns represented by the consensus belief $\mathcal{B}_a$ does not change with the entry of a risk-free asset. Generally, the tangency relation does not hold in a market populated by heterogeneous investors.

7.3. **Derivation of the ZHCAPM.** In this subsection, we show that one can still derive the ZHCAPM relation, which is the *zero-beta heterogenous Capital Asset Pricing Model*. The derivation actually becomes simpler under the new set up.

Proposition 7.2. *Define the expected market return under the consensus belief as*

$$\mathbb{E}_a(\tilde{r}_m) = \mathbb{E}_a(\frac{\tilde{W}_m}{W_{m0}} - 1)$$

, where $W_{m0} = \boldsymbol{\pi}_m^T\mathbf{1}$ is the total initial market wealth.

In equilibrium, the Zero-beta CAPM relation can be expressed as

$$\mathbb{E}_a[\tilde{\mathbf{r}}] - (\lambda_a - 1)\mathbf{1} = \boldsymbol{\beta}[\mathbb{E}_a(\tilde{r}_m) - (\lambda_a - 1)], \tag{7.11}$$

where

$$\boldsymbol{\beta} = (\beta_1, \beta_2, \cdots, \beta_N)^T, \qquad \beta_j = \frac{Cov_a(\tilde{r}_m, \tilde{r}_j)}{\sigma_a^2(\tilde{r}_m)}, \qquad j = 1, \cdots, N,$$

Proof. Let

$$\mu_{am} = \mathbb{E}_a(\frac{\tilde{W}_m}{W_{m0}}) = \mathbb{E}_a(\tilde{r}_m) + 1 \qquad \text{and} \qquad \boldsymbol{\omega}_m = \frac{\boldsymbol{\pi}_m}{W_{m0}}$$

From the fact that $\tilde{W}_m = \boldsymbol{\pi}_m^T(\tilde{\mathbf{r}} + \mathbf{1})$ and $\boldsymbol{\omega}_m^T \mathbf{1} = 1$, it follows that

$$\begin{aligned}\mu_{am} - \lambda_a &= \boldsymbol{\omega}_m^T(\boldsymbol{\mu}_a - \lambda_a \mathbf{1}) = \boldsymbol{\omega}_m^T \left(\frac{1}{I} \Theta^{-1} V_a \boldsymbol{\pi}_m \right) \\ &= \frac{W_{m0}}{I} \Theta^{-1} \sigma_{am}^2 \end{aligned} \tag{7.12}$$

Also we can rearrange equation (7.10) to get

$$\boldsymbol{\mu}_a - \lambda_a \mathbf{1} = \frac{1}{I} \Theta^{-1} V_a \boldsymbol{\pi}_m \tag{7.13}$$

which can be written for each asset j as follows

$$\mu_{a,j} - \lambda_a = \frac{W_{m0}}{I} \Theta^{-1} \sum_{k=1}^{N} \sigma_{a,jk} \omega_{mk} = \frac{W_{m0}}{I} \Theta^{-1} Cov_a(\tilde{r}_j, \tilde{r}_m) \tag{7.14}$$

Therefore

$$\frac{\mu_{a,j} - \lambda_a}{\mu_{am} - \lambda_a} = \frac{Cov_a(\tilde{r}_j, \tilde{r}_m)}{\sigma_{am}^2} = \beta_j \tag{7.15}$$

Equation (7.15) leads to

$$\boldsymbol{\mu}_a - \lambda_a \mathbf{1} = \boldsymbol{\beta}(\mu_{am} - \lambda_a) \tag{7.16}$$

(7.16) leads to the relation in (7.11), which completes the proof.

□

8. CONCLUSION

In this paper, we have showed that aggregation of the market through the construction of a consensus belief is still feasible without the assumption of the existence of a risk-free asset in the market, and it is a direct generalization of the case with the assumption of a risk-free asset. This has allowed to study the impact of heterogeneity from different sources in a market without assuming the existence of a risk-free asset and compare the results to the case in which the risk-free assumption is made, also explanation for some empirical findings in the market are provided. In principle, we showed that the market aggregate beliefs and the equilibrium prices are still a weighted average of individual beliefs and risk aversions even without assuming a risk-free asset and the zero-beta relation in the traditional finance theory still hold in a heterogeneous market. In terms of market portfolio frontiers in the mean-standard deviation space, under our model, we showed that the market portfolio remains on the efficient frontier while individual optimal portfolio might not be on it, also the tangency relation does not hold in general with heterogeneous beliefs in the market.

This paper provides a general yet simple framework for market aggregation and showed that the existence of a risk-free asset is not a necessary assumption. The next step will be to extend the model to a dynamic setting and allow investors to learn overtime from the new information in the market, this can be done using the Bayesian updating rule. This will give us a richer modelling environment and hopefully lead to a better understanding of the phenomenons in our financial market, both present and in the past. This is the direction of our future research.

APPENDIX A. PROOF OF THE INEQUALITY IN EQUATION (5.29)

Assume there are 2 investors in the market, ie $I = 2$. We have from equation 3.5 and 3.3 that

$$\Omega_a^{-1} = \frac{\theta_1^{-1}}{\theta_1^{-1} + \theta_2^{-2}} \Omega_1^{-1} + \frac{\theta_2^{-1}}{\theta_1^{-1} + \theta_2^{-2}} \Omega_2^{-1} \tag{A.1}$$

Since Ω_1^{-1} is symmetric, which implies that $\Omega_1^{-1} = P^T P$, where P is a square matrix and $\Omega_2^{-1} = P^T(Q^T D Q)P$, where Q is an orthogonal matrix and D is a diagonal matrix [13].

Therefore (5.29) can be re-written as

$$\Omega_a^{-1} = P^T \left(\frac{\theta_1^{-1}}{\theta_1^{-1} + \theta_2^{-2}} I + \frac{\theta_2^{-1}}{\theta_1^{-1} + \theta_2^{-2}} Q^T D Q \right) P$$

due to the orthogonality of Q

$$= P^T Q^T \left(\frac{\theta_1^{-1}}{\theta_1^{-1} + \theta_2^{-2}} I + \frac{\theta_2^{-1}}{\theta_1^{-1} + \theta_2^{-2}} D \right) QP$$

$$= (QP)^T \left(\frac{\theta_1^{-1}}{\theta_1^{-1} + \theta_2^{-2}} I + \frac{\theta_2^{-1}}{\theta_1^{-1} + \theta_2^{-2}} D \right) (QP) \tag{A.2}$$

So

$$\Omega_a = (QP)^{-1} \left(\frac{\theta_1^{-1}}{\theta_1^{-1} + \theta_2^{-2}} I + \frac{\theta_2^{-1}}{\theta_1^{-1} + \theta_2^{-2}} D \right) ((QP)^T)^{-1}$$

$$= (QP)^{-1} \Lambda_1 ((QP)^T)^{-1} \tag{A.3}$$

where

$$\Lambda_1 = \left(\frac{\theta_1^{-1}}{\theta_1^{-1} + \theta_2^{-2}} I + \frac{\theta_2^{-1}}{\theta_1^{-1} + \theta_2^{-2}} D \right)^{-1}$$

With $I = 2$

$$\bar{\Omega} = \frac{\theta_1^{-1}}{\theta_1^{-1} + \theta_2^{-2}} \Omega_1 + \frac{\theta_2^{-1}}{\theta_1^{-1} + \theta_2^{-2}} \Omega_2 \tag{A.4}$$

[13] since Ω_1^{-1} symmetric, so we can write $\Omega_1^{-1} = P^T A P$ for some symmetric matrix A and A being symmetric means that $A = Q^T D Q$ for some diagonal matrix D (Q orthogonally diagonalizes A) according to the *Spectral Decomposition Theorem*

but we had $\Omega_1^{-1} = P^T P \quad \Rightarrow \quad \Omega_1 = (P^T Q^T Q P)^{-1} = (QP)^{-1} I ((QP)^T)^{-1}$

and $\Omega_2^{-1} = (QP)^T D (QP) \quad \Rightarrow \quad \Omega_2 = (QP)^{-1} D^{-1} ((QP)^T)^{-1}$

So

$$\begin{aligned}\bar{\Omega} &= \frac{\theta_1^{-1}}{\theta_1^{-1} + \theta_2^{-2}} \Big((QP)^{-1} I ((QP)^T)^{-1} \Big) + \frac{\theta_2^{-1}}{\theta_1^{-1} + \theta_2^{-2}} \Big((QP)^{-1} D^{-1} ((QP)^T)^{-1} \Big) \\ &= (QP)^{-1} \Big(\frac{\theta_1^{-1}}{\theta_1^{-1} + \theta_2^{-2}} I + \frac{\theta_2^{-1}}{\theta_1^{-1} + \theta_2^{-2}} D^{-1} \Big) ((QP)^T)^{-1} \\ &= (QP)^{-1} \Lambda_2 ((QP)^T)^{-1} \end{aligned} \tag{A.5}$$

where

$$\Lambda_2 = \frac{\theta_1^{-1}}{\theta_1^{-1} + \theta_2^{-2}} I + \frac{\theta_2^{-1}}{\theta_1^{-1} + \theta_2^{-2}} D^{-1}$$

(A.3)-(A.5) leads to

$$\bar{\Omega} - \Omega_a = (QP)^{-1} (\Lambda_2 - \Lambda_1) ((QP)^T)^{-1} \tag{A.6}$$

Now

$$\Lambda_2 - \Lambda_1 = (\gamma_1 I + \gamma_2 D^{-1}) - (\gamma_1 I + \gamma_2 D)^{-1} \tag{A.7}$$

where $\gamma_1 = \frac{\theta_1^{-1}}{\theta_1^{-1} + \theta_2^{-2}}$ and $\gamma_2 = \frac{\theta_2^{-1}}{\theta_1^{-1} + \theta_2^{-2}} \Rightarrow \gamma_1 + \gamma_2 = 1$.

Denotes d_i as the diagonal elements of D for $i = 1, 2, \cdots, N$, we expand equation (A.7) as

$$\Lambda_2 - \Lambda_1 = \begin{pmatrix} \gamma_1 + \gamma_2 d_1^{-1} & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & \gamma_1 + \gamma_2 d_N^{-1} \end{pmatrix} - \begin{pmatrix} (\gamma_1 + \gamma_2 d_1)^{-1} & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & (\gamma_1 + \gamma_2 d_N)^{-1} \end{pmatrix} \tag{A.8}$$

so for the i^{th} diagonal element ie.

$$\gamma_1 + \gamma_2 d_i^{-1} - (\gamma_1 + \gamma_2 d_i)^{-1} = \frac{(\gamma_1 + \gamma_2 d_i^{-1})(\gamma_1 + \gamma_2 d_i) - 1}{(\gamma_1 + \gamma_2 d_i)} \tag{A.9}$$

the nominator of the above equation is

$$(\gamma_1^2 + \gamma_2^2 - 1) + \gamma_1\gamma_2(d_i + d_i^{-1}) \tag{A.10}$$

Now for $f(x) = x + 1/x$, $\min(f(x)) = 2$, so

$$\begin{aligned}(\gamma_1^2 + \gamma_2^2 - 1) + \gamma_1\gamma_2(d_i + d_i^{-1}) &\geq ((\gamma_1^2 + \gamma_2^2 - 1) + 2\gamma_1\gamma_2 \\ &= (\gamma_1 + \gamma_2)^2 - 1 = 1^2 - 1 = 0 \end{aligned} \tag{A.11}$$

Hence $\Lambda_1 - \Lambda_2$ is a semi-positive definite matrix since $\gamma_1 + \gamma_2 d_i > 0$ for all i [14].

Now $((QP)^{-1})^T = ((QP)^T)^{-1}$, so $(QP)^{-1}$ diagonalizes $\bar(\Omega) - \Omega_a$, and since the diagonal elements of $\Lambda_2 - \Lambda_1$ are all non-negative, the eigenvalues of $\bar{\Omega} - \Omega_a$ are non-negative, hence $\bar{\Omega} - \Omega_a$ is semi-positive definite. We are done.

[14] since $\Omega_2^{-1} = (QP)^T D(QP)$ and Ω_2^{-1} is semi-positive definite and so the eigenvalues must be non-negative, hence $d_i \geq 0$.

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