HETEROGENEITY, CONVERGENCY, AND AUTOCORRELATIONS

XUE-ZHONG HE* AND YOUWEI LI**

*School of Finance and Economics

University of Technology, Sydney

PO Box 123 Broadway

NSW 2007, Australia

and

**School of Management and Economics

Queen's University Belfast

25 University Square, BT7 1NN, Belfast, UK

ABSTRACT. This paper contributes to the development of the recent literature on the explanation power and calibration issue of heterogeneous asset pricing models by presenting a simple stochastic market fraction asset pricing model of two types of traders (fundamentalists and trend followers) under a market maker scenario. It seeks to explain aspects of financial market behavior (such as market dominance, convergence of the market price to the fundamental price, and under- and over-reaction) and to characterize various statistical properties (including the convergence of the limiting distribution and autocorrelation structure) of the stochastic model by using the dynamics of the underlying deterministic system, traders' heterogeneous behavior and market fractions. A statistical analysis based on Monte Carlo simulations shows that the long-run behavior, convergence of the market prices to the fundamental price, limiting distributions, and various under and over-reaction autocorrelation patterns of returns can be characterized by the stability and bifurcations of the underlying deterministic system. Our analysis underpins the mechanisms on various market behaviors (such as under/over-reactions), market dominance and stylized facts in high frequency financial markets.

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Corresponding author: Xuezhong (Tony) He, School of Finance and Economics, University of Technology, Sydney, PO Box 123 Broadway, NSW 2007, Australia. Email: Tony.He1@uts.edu.au. Ph: (61 2) 9514 7726. Fax: (61 2) 9514 7711.

1. INTRODUCTION

Traditional economic and finance theory is based on the assumptions of investor homogeneity and the efficient market hypothesis. However, there is a growing dissatisfaction with models of asset price dynamics, based on the representative agent paradigm, as expressed for example by Kirman (1992), and the extreme informational assumptions of rational expectations. As a result, the literature has seen a rapidly increasing number of heterogeneous agents models, see recent survey papers by Hommes (2006) and LeBaron (2006). These models characterize the dynamics of financial asset prices; resulting from the interaction of heterogeneous agents having different attitudes to risk and having different expectations about the future evolution of prices.[1] For example, Brock and Hommes (1997, 1998) proposed a simple *Adaptive Belief System* to model economic and financial markets. Agents' decisions are based upon predictions of future values of endogenous variables whose actual values are determined by the equilibrium equations. A key aspect of these models is that they exhibit feedback of expectations. Agents adapt their beliefs over time by choosing from different predictors or expectations functions, based upon their past performance as measured by the realized profits. The resulting dynamical system is nonlinear and, as Brock and Hommes (1998) show, capable of generating the entire *zoo* of complex behavior from local stability to high order cycles and even chaos as various key parameters of the model change. It has been shown (e.g. Hommes (2002)) that such simple nonlinear adaptive models are capable of explaining important empirical observations, including fat tails, clustering in volatility and long memory of real financial series. The analysis of the stylized simple evolutionary adaptive system, and its numerical analysis provides insight into the connection between individual and market behavior. Specifically, it provides insight into whether asset prices in real markets are driven only by news or, are at least in part, driven by market psychology.

The heterogeneous agents literature attempts to address two interesting issues among many others. It attempts to explain various types of market behavior, and to replicate the well documented empirical findings of actual financial markets, the stylized facts. The recent literature has demonstrated the ability to explain various types of market behavior. However, in relation

[1]See, e.g., Arthur *et al.* (1997), Brock and Hommes (1997, 2002), Brock and LeBaron (1996), Bullard and Duffy (1999), Chen and Yeh (1997, 2002), Chiarella (1992), Chiarella *et al.* (2002), Chiarella and He (2001, 2002, 2003*b*), Dacorogna *et al.* (1995), Day and Huang (1990), De Long et al (1990), Farmer and Joshi (2002), Frankel and Froot (1987), Gaunersdorfer (2000), Hommes (2001, 2002), Iori (2002), LeBaron (2000, 2001, 2002), LeBaron *et al.* (1999), Lux (1995, 1997, 1998) and Lux and Marchesi (1999))

to the stylized facts, there is a gap between the heterogeneous agents models and observed empirical findings. It is well known that most of the stylized facts can be observed only for high frequency data (e.g. daily) and not for low frequency data (e.g. yearly). However, two unrealistic assumptions underpin this literature.[2] The first is a risk-free rate of approximately 10 per-cent per trading period.[3] Given that this rate is crucial for model calibration in generating stylized facts[4], it is obviously unrealistic. Second, the unrealistic nature of the assumed trading period is problematic for the quantitative calibration to actual time series. As pointed out by LeBaron (2002), *'This (unrealistic trading period) is fine for early qualitative comparisons with stylized facts, but it is a problem for quantitative calibration to actual time series'*.

Another more important issue for various heterogeneous asset pricing models is the interplay of noisy and deterministic dynamics. Given that deterministic models are simplified versions of realistic stochastic models and stability and bifurcation are the most powerful tools (among other things) to investigate the dynamics of nonlinear system, it is interesting to know how deterministic properties influence the statistical properties, such as the existence and convergence of stationary process, and the autocorrelation (AC) structure of the corresponding stochastic system. In particular, we can ask if there is a connection between different types of attractors and bifurcations of the underlying deterministic skeleton and various invariant measures, and AC patterns of the stochastic system, respectively. This has the potential to provide insights into the mechanisms of generating various invariant measures, AC patterns and stylized facts in financial markets. These issues are investigated in a context of a simple heterogeneous asset pricing model in this paper. At present, the mathematic theory has not yet been able to achieve these tasks in general. Consequently, statistical analysis and Monte Carlo simulations is the approach adopted in this paper.

[2]See, e.g., Arthur *et al.* (1997), Brock and Hommes (1997), Chen and Yeh (2002), Chiarella et al (2002), Chiarella and He (2002, 2003*b*), Iori (2002), LeBaron (2002), LeBaron *et al.* (1999), Levy *et al.* (1994)).
[3]Apart from $r_f = 1\%$ in Gaunersdorfer (2000) and LeBaron (2001) and $r_f = 0.04\%$ in Hommes (2002).
[4]In this literature, as risk-free rate of trading period decreases, demand on the risky asset increases. Consequently, the price of the risky asset become rather larger numbers resulting sometimes in break-down in theoretic analysis and overflows in numerical simulations. In addition, some of interesting dynamics disappear as the risk-free rate of trading period decreases to realistic level (e.g. (5/250)% per day given a risk-free rate of 5% p.a. and 250 trading days per year).

This paper builds upon the existent literature by incorporating a realistic trading period[5], which eliminates the unrealistic risk-free rate assumption, whilst also introducing market fractions of heterogeneous traders into a simple asset-pricing model. In this paper this model is referred to as the Market Fraction (**MF**) Model. The model assumes three types of participants in the asset market. This includes two groups of boundedly rational traders—fundamentalists (also called informed traders) and trend followers (also called less informed traders or chartists), and a market-maker. The aim of this paper is to show that in the MF model the long-run behavior of asset prices and the autocorrelation structure of the stochastic system can be characterized by the dynamics of the underlying deterministic system, traders' behavior, and market fractions. In addition, this paper also contributes to the literature how to use statistical analysis based on Monte Carlo simulations to study the interplay of noise and deterministic dynamics in the context of heterogeneous asset pricing models. The statistical analysis shows that the long-run behavior and convergence of the market prices, and various under- and over-reaction AC patterns of returns can be characterized by the stability and bifurcations of the underlying deterministic system. Our analysis gives us some insights into the mechanism of various market behavior (such as under/over-reactions), market dominance, and stylized facts in high frequency financial markets.

This paper is organized as follows. Section 2 outlines a market fraction model of heterogeneous agents with the market clearing price set by a market maker, introduces the expectations function and learning mechanisms of the fundamentalists and trend followers, and derives a full market fraction model on asset price dynamics. Price dynamics of the underlying deterministic model is examined in Section 3. Statistical analysis, based on Monte Carlo simulations, of the stochastic model is given in Section 4. By using the concept of a random fixed point, we examine the long-run behavior and convergence of the market price to the fundamental price and to an invariant measure. By choosing different sets of parameters near different types of bifurcation boundaries of the underlying deterministic system, we explore various under and over-reaction AC patterns. Section 5 concludes and all proofs and additional statistical results are included in the Appendices.

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[5]In fact, the trading period of the model can be scaled to any level of trading frequency ranging from annually, monthly, weekly, to daily. However, we focus on a daily trading period (i.e. $K = 250$) in this paper.

2. Heterogeneous Beliefs, Market Fractions and Market-Maker

Both empirical and theoretical studies show that market fractions among different types of traders have an important role to play in financial markets. Empirical evidence from Taylor and Allen (1992) suggests that at least 90% of the traders place some weights on technical analysis at one or more time horizons. In particular, traders rely more on technical analysis, as opposed to the fundamental analysis, at shorter time horizons. As the length of time horizons increases, more traders rely on the fundamental rather than technical analysis. In addition, there is a certain proportion of traders who do not change their strategies over all time horizons. Theoretically, the study by Brock and Hommes (1997) shows that, when different groups of traders, such as fundamentalists and chartists, having different expectations about future prices and dividends compete between trading strategies and choose their strategy according to an evolutionary *fitness measure*, the corresponding deterministic system exhibits rational routes to randomness. The adaptive switching mechanism proposed by Brock and Hommes (1997) is an important element of the adaptive belief model. It is based on both a *fitness function* and a discrete choice probability. In this paper, we take a simplified version of the Brock and Hommes' framework. The MF model assumes that the market fractions among heterogeneous agents are fixed parameters. Apart from mathematical tractability, this simplification is motivated as follows. First, because of the amplifying effect of the exponential function used in the discrete choice probability, the market fractions become very sensitive to price changes and the fitness functions. Therefore, it is not very clear to see how different group of traders do actually influence the market price. Secondly, when agents switch intensively, it becomes difficult to characterize market dominance when dealing with heterogeneous trading strategies. Thirdly, it is important to understand how the behaviors of different types of agents are linked to certain dynamics (such as the autocorrelation structure we discuss later). Such an analysis becomes clear when we isolate the market fractions from switching. In doing so, we can examine explicitly the influence of the market fractions on the price behavior.

The set up follows the standard discounted value asset pricing model with heterogeneous agents, which is closely related to the framework of Day and Huang (1990), Brock and Hommes (1997, 1998) and Chiarella and He (2002, 2003*b*). The market clearing price is arrived at via a market maker scenario rather than the Walrasian scenario. We focus on a simple case in which there are three classes of participants in the asset market: two groups of traders, fundamentalists and trend followers, and a market maker, as described in the following discussion.

2.1. **Market Fractions and Market Clearing Price under a Market Maker.** Consider an asset pricing model with one risky asset and one risk free asset. It is assumed that the risk free asset is perfectly elastically supplied at a gross return of $R = 1 + r/K$, where r stands for a constant risk-free rate per annum and K stands for the trading frequency measured in a year. Typically, $K = 1, 12, 52$ and 250 for of trading period of a year, a month, a week and a day, respectively. To focus on the stylized facts observed from daily price movement in financial markets, we select $K = 250$ in our following discussion.

Let P_t be the (ex dividend) price per share of the risky asset at time t and $\{D_t\}$ be the stochastic dividend process of the risky asset. Then the wealth of a typical trader-h at $t+1$ is given by

$$W_{h,t+1} = RW_{h,t} + [P_{t+1} + D_{t+1} - RP_t]z_{h,t}, \tag{2.1}$$

where $W_{h,t}$ and $z_{h,t}$ are the wealth and the number of shares of the risky asset purchased by trader-h at t, respectively. Let $E_{h,t}$ and $V_{h,t}$ be the *beliefs* of type h traders about the conditional expectation and variance of quantities at $t+1$ based on their information set at time t. Denote by R_{t+1} the excess capital gain on the risky asset at $t+1$, that is

$$R_{t+1} = P_{t+1} + D_{t+1} - R\,P_t. \tag{2.2}$$

Then it follows from (2.1) and (2.2) that

$$E_{h,t}(W_{t+1}) = RW_t + E_{h,t}(R_{t+1})z_{h,t}, \qquad V_{h,t}(W_{t+1}) = z_{h,t}^2 V_{h,t}(R_{t+1}). \tag{2.3}$$

Assume that trader-h has a constant absolute risk aversion (CARA) utility function with the risk aversion coefficient a_h (e.g. $U_h(W) = -e^{-a_h W}$). By expected utility maximization, trader-h's optimal demand on the risky asset $z_{h,t}$ is given by

$$z_{h,t} = \frac{E_{h,t}(R_{t+1})}{a_h V_{h,t}(R_{t+1})}. \tag{2.4}$$

Given the heterogeneity and the nature of asymmetric information among traders, we consider two most popular trading strategies corresponding to two types of boundedly rational traders—fundamentalists and trend followers, and their beliefs will be defined in the following discussion. Assume the market fraction of the fundamentalists and trend followers is n_1 and n_2 with risk aversion coefficient a_1 and a_2, respectively. Let $m = n_1 - n_2 \in [-1, 1]$. Obviously, $m = 1$ and -1 correspond to the cases when all the traders are fundamentalists or trend followers, respectively. Assume a zero supply of outside shares. Then, using (2.4), the aggregate excess demand per trader ($z_{e,t}$) is given by

$$z_{e,t} \equiv n_1 z_{1,t} + n_2 z_{2,t} = \frac{1+m}{2} \frac{E_{1,t}[R_{t+1}]}{a_1 V_{1,t}[R_{t+1}]} + \frac{1-m}{2} \frac{E_{2,t}[R_{t+1}]}{a_2 V_{2,t}[R_{t+1}]}. \tag{2.5}$$

To complete the model, we assume that the market is cleared by a market maker. The role of the market maker is to take a long (when $z_{e,t} < 0$) or short (when $z_{e,t} > 0$) position so as to clear the market. At the end of period t, after the market maker has carried out all transactions, he or she adjusts the price for the next period in the direction of the observed excess demand. Let μ be the speed of price adjustment of the market maker (this can also be interpreted as the market aggregate risk tolerance). To capture unexpected market news or noise created by *noise traders*, we introduce a noisy demand term $\tilde{\delta}_t$ which is an i.i.d. normally distributed random variable[6] with $\tilde{\delta}_t \sim \mathcal{N}(0, \sigma_\delta^2)$. Based on these assumptions, the market price is determined by

$$P_{t+1} = P_t + \mu z_{e,t} + \tilde{\delta}_t.$$

From (2.5), this becomes

$$P_{t+1} = P_t + \frac{\mu}{2}\left[(1+m)\frac{E_{1,t}[R_{t+1}]}{a_1 V_{1,t}[R_{t+1}]} + (1-m)\frac{E_{2,t}[R_{t+1}]}{a_1 V_{2,t}[R_{t+1}]}\right] + \tilde{\delta}_t. \tag{2.6}$$

It should be pointed out that the market maker behavior in this model is highly stylized. For instance, the inventory of the market maker built up as a result of the accumulation of various long and short positions is not considered. This could affect his or her behavior and the market maker price setting role in (2.6) could be a function of the inventory. Allowing μ to be a function of inventory would be one way to model such behavior. We should also seek to explore the micro-foundations of the coefficient μ. Such considerations are left to future research.

[6] In this paper, we assume a constant volatility noisy demand and the volatility is related to an average fundamental price level. This noisy demand may also depend on the market price. Theoretically, how the price dynamics are influenced by adding different noisy demand is still a difficult problem. Here, we focus on the constant volatility noisy demand case and use Monte Carlo simulations and statistical analysis to gain some insights into this problem.

2.2. **Fundamentalists.** Denote by $F_t = \{P_t, P_{t-1}, \cdots; D_t, D_{t-1}, \cdots\}$ the common information set formed at time t. We assume that, apart from the common information set, the fundamentalists have *superior* information on the fundamental value, P_t^*, of the risky asset, which is assumed to follow a stationary random walk process[7]

$$P_{t+1}^* = P_t^*[1 + \sigma_\epsilon \tilde{\epsilon}_t], \qquad \tilde{\epsilon}_t \sim \mathcal{N}(0, 1), \qquad \sigma_\epsilon \geq 0, \qquad P_0^* = \bar{P} > 0, \tag{2.7}$$

where $\tilde{\epsilon}_t$ is independent of the noisy demand process $\tilde{\delta}_t$. This specification ensures that neither fat tails nor volatility clustering are brought about by the fundamental price process. Hence, emergence of any autocorrelation pattern of the return of the risky asset in our late discussion would be driven by the trading process itself.

For the fundamentalists, because they realize the existence of non-fundamental traders, such as trend followers to be introduced in the following discussion, they believe that the stock price may be driven away from the fundamental value. More precisely, we assume that the conditional mean and variance of the fundamental traders are, respectively

$$E_{1,t}(P_{t+1}) = P_t + \alpha(P_{t+1}^* - P_t), \qquad V_{1,t}(P_{t+1}) = \sigma_1^2, \tag{2.8}$$

where σ_1^2 is a constant, and $\alpha \in [0, 1]$ is the weight on the fundamental price which measures the speed of price adjustment of the fundamentalists toward the fundamental value. That is, the expected price of the fundamentalists is a weighted average of the fundamental price and the latest market price, while the variance of the price is a constant. In general, the fundamental traders believe that markets are efficient and prices converge to the fundamental value. A high (low) weight of α leads to a quick (slow) adjustment of their expected price towards the fundamental price.

2.3. **Trend followers.** Unlike the fundamentalists, trend followers are technical traders who believe the future price change can be predicted from various patterns or trends generated from the history of prices. The trend followers are assumed to extrapolate the latest observed price change over prices' long-run sample mean and to adjust their variance estimate accordingly. More precisely, their conditional mean and variance are assumed to satisfy

$$E_{2,t}(P_{t+1}) = P_t + \gamma(P_t - u_t), \qquad V_{2,t}(P_{t+1}) = \sigma_1^2 + b_2 v_t, \tag{2.9}$$

[7]As we know that the fundamental value driven by this random walk process can be negative.

where $\gamma, b_2 \geq 0$ are constants, and u_t and v_t are sample mean and variance, respectively, which may follow some learning processes. The parameter γ measures the extrapolation rate and high (low) values of γ correspond to strong (weak) extrapolation from the trend followers. The coefficient b_2 measures the influence of the sample variance on the conditional variance estimated by the trend followers who believe in a more volatile price movement. Various learning schemes[8] can be used to estimate the sample mean u_t and variance v_t. In this paper we assume that

$$u_t = \delta u_{t-1} + (1-\delta)P_t, \tag{2.10}$$

$$v_t = \delta v_{t-1} + \delta(1-\delta)(P_t - u_{t-1})^2, \tag{2.11}$$

where $\delta \in [0,1]$ is a constant. This is a limiting *geometric decay process* when the memory lag length tends to infinity.[9] Basically, a geometric decay probability process $(1-\delta)\{1, \delta, \delta^2, \cdots\}$ is associated to the historical prices $\{P_t, P_{t-1}, P_{t-2}, \cdots\}$. The parameter δ measures the geometric decay rate. For $\delta = 0$, the sample mean $u_t = P_t$, which is the latest observed price, while $\delta = 0.1, 0.5, 0.95$ and 0.999 gives a half life of 0.43 day, 1 day, 2.5 weeks and 2.7 years, respectively. The selection of this process is two fold. First, traders tend to put a higher weight to the most recent prices and lesser weight to the more remote prices when they estimate the sample mean and variance. Secondly, we believe that this geometric decay process may contribute to certain autocorrelation patterns, even the long memory feature observed in real financial markets. In addition, it has the mathematical advantage of analytical tractability.

2.4. **The Complete Stochastic Model.** To simplify our analysis, we assume that the dividend process D_t follows a normal distribution $D_t \sim \mathcal{N}(\bar{D}, \sigma_D^2)$, the expected long-run fundamental value $\bar{P} = \bar{D}/(R-1)$, and the unconditional variances of price and dividend over the trading

[8]For related studies on heterogeneous learning in asset pricing models with heterogeneous agents who's conditional mean and variance follow various learning processes, we refer to Chiarella and He (2003*a*, 2004).
[9]See Chiarella *et. al.* (2006) for the proof.

period are related by $\sigma_D^2 = q\sigma_1^2$.[10] Based on assumptions (2.8)-(2.9),

$$E_{1,t}(R_{t+1}) = P_t + \alpha(P^*_{t+1} - P_t) + \bar{D} - R\,P_t = \alpha(P^*_{t+1} - P_t) - (R-1)(P_t - \bar{P}),$$

$$V_{1,t}(R_{t+1}) = (1+q)\sigma_1^2$$

and hence the optimal demand for the fundamentalists is given by

$$z_{1,t} = \frac{1}{a_1(1+q)\sigma_1^2}[\alpha(P^*_{t+1} - P_t) - (R-1)(P_t - \bar{P})]. \tag{2.12}$$

In particular, when $P^*_t = \bar{P}$,

$$z_{1,t} = \frac{(\alpha + R - 1)(\bar{P} - P_t)}{a_1(1+q)\sigma_1^2}. \tag{2.13}$$

Similarly, from (2.9), (using $\bar{D} = (R-1)\bar{P}$)

$$E_{2,t}(R_{t+1}) = P_t + \gamma(P_t - u_t) + \bar{D} - R\,P_t = \gamma(P_t - u_t) - (R-1)(P_t - \bar{P}),$$

$$V_{2,t}(R_{t+1}) = \sigma_1^2(1 + q + b\,v_t),$$

where $b = b_2/\sigma_1^2$. Hence the optimal demand of the trend followers is given by

$$z_{2,t} = \frac{\gamma(P_t - u_t) - (R-1)(P_t - \bar{P})}{a_2\sigma_1^2(1 + q + b\,v_t)}. \tag{2.14}$$

Subsisting (2.12) and (2.14) into (2.6), the price dynamics under a market maker is determined by the following 4-dimensional stochastic difference system (**SDS** hereafter)

$$\begin{cases} P_{t+1} = P_t + \dfrac{\mu}{2}\left[\dfrac{1+m}{a_1(1+q)\sigma_1^2}[\alpha(P^*_{t+1} - P_t) - (R-1)(P_t - \bar{P})] \right. \\ \qquad\qquad \left. + (1-m)\dfrac{\gamma(P_t - u_t) - (R-1)(P_t - \bar{P})}{a_2\sigma_1^2(1 + q + b\,v_t)}\right] + \tilde{\delta}_t, \\ u_t = \delta u_{t-1} + (1-\delta)P_t, \\ v_t = \delta v_{t-1} + \delta(1-\delta)(P_t - u_{t-1})^2, \\ P^*_{t+1} = P^*_t[1 + \sigma_\epsilon \tilde{\epsilon}_t]. \end{cases} \tag{2.15}$$

[10] In this paper, we choose $\sigma_1^2 = \sigma_{\bar{P}}^2/K$ and $q = r^2$. This can be justified as follows. Let $\sigma_{\bar{P}}$ be the annual volatility of P_t and $\bar{D}_t = rP_t$ be the annual dividend. Then the annual variance of the dividend $\bar{\sigma}_D^2 = r^2\sigma_{\bar{P}}^2$. Therefore $\sigma_D^2 = \bar{\sigma}_D^2/K = r^2\sigma_{\bar{P}}^2/K = r^2\sigma_1^2$. For all numerical simulations in this paper, we choose $\bar{P} = \$100, r = 5\%$ p.a. $\sigma = 20\%$ p.a., $K = 250$. Correspondingly, $R = 1 + 0.05/250 = 1.0002, \sigma_1^2 = (100 \times 0.2)^2/250 = 8/5$ and $\sigma_D^2 = 1/250$.

It has been widely accepted that stability and bifurcation theory is a powerful tool in the study of asset-pricing dynamics (see, for example, Day and Huang (1990), Brock and Hommes (1997, 1998) and Chiarella and He (2002, 2003*b*)). However, the question how the stability and various types of bifurcation of the underlying deterministic system affect the nature of the stochastic system, including stationarity, distribution and statistic properties of returns, is not very clear at the current stage. Although the techniques discussed in Arnold (1998) may be useful in this regard, the mathematical analysis of nonlinear stochastic dynamical system is still difficult in general. In this paper, we consider first the corresponding deterministic skeleton of the stochastic model by assuming that the fundamental price is given by its long-run value $P_t^* = \bar{P}$ and there is no demand shocks, i.e. $\sigma_\delta = \sigma_\epsilon = 0$. We then conduct a stochastic analysis of the stochastic model through Monte Carlo simulation.

3. Dynamics of the Deterministic Model

When the long run fundamental price is a constant and there is no noisy demand, the 4-dimensional stochastic system (2.15) reduces to the following 3-dimensional deterministic difference system (**DDS** hereafter)

$$\begin{cases} P_{t+1} = P_t + \mu \dfrac{1+m}{2}\left[\dfrac{(1-\alpha-R)(P_t-\bar{P})}{a_1(1+q)\sigma_1^2}\right] + \dfrac{1-m}{2}\left[\dfrac{\gamma(P_t-u_t)-(R-1)(P_t-\bar{P})}{a_2\sigma_1^2(1+q+b\,v_t)}\right], \\ u_t = \delta u_{t-1} + (1-\delta)P_t, \\ v_t = \delta v_{t-1} + \delta(1-\delta)(P_t-u_{t-1})^2. \end{cases} \tag{3.1}$$

The following result on the existence and uniqueness of steady state of the deterministic system is obtained.

Proposition 3.1. *For DDS (3.1),* $(P_t, u_t, v_t) = (\bar{P}, \bar{P}, 0)$ *is the unique steady state.*

Proof. See Appendix A.1. □

We call this unique steady state the fundamental steady state. In the following discussion, we focus on the stability and bifurcation of the fundamental steady state of the deterministic model. We first examine two special cases $m = 1$ and $m = -1$, before we deal with the general case $m \in (-1, 1)$.

3.1. **The case $m = 1$.** In this case, the following result on the global stability and bifurcation is obtained.

Proposition 3.2. *For DDS (3.1), if all the traders are fundamentalists, i.e. $m = 1$, then the fundamental price $\bar{P}$ is globally asymptotically stable if and only if*

$$0 < \mu < \mu_{0,1} \equiv \frac{2a_1(1+q)\sigma_1^2}{(R+\alpha-1)}. \tag{3.2}$$

In addition, $\mu = \mu_{0,1}$ leads to a flip bifurcation with $\lambda = -1$, where

$$\lambda = 1 - \mu\frac{R+\alpha-1}{a_1(1+q)\sigma_1^2}. \tag{3.3}$$

Proof. See Appendix A.2. □

The stability region of the fundamental price $\bar{P}$ is plotted in (α, μ) plane in Fig.A.1 in Appendix A.2, where $\mu_{0,1}(1) = [2a_1(1+q)\sigma_1^2]/R$ for $\alpha = 1$ and $\mu_{0,1}(0) = [2a_1(1+q)\sigma_1^2]/(R-1)$ for $\alpha = 0$. The stability condition (3.2) is equivalent to $\mu(R+\alpha-1) < 2a_1(1+q)\sigma_1^2$, implying that the fundamental price is locally stable as long as the reactions from both the market maker and the fundamentalists are balanced (i.e. a high (low) μ is balanced by a low (high) α so that the product $\mu(R+\alpha-1)$ is below the constant $2a_1(1+q)\sigma_1^2$). Given the stabilizing role (to the fundamental price) of the fundamentalists, over-reactions from either the fundamentalists or the market maker will push the market price to flipping around the fundamental price. Numerical simulations indicate that the over-reaction from either the market maker or the fundamentalists can push the price to explode (through the flip bifurcation).

3.2. **The case $m = -1$.** Similarly, we obtain the following stability and bifurcation result when all traders are trend followers.

Proposition 3.3. *For DDS (3.1), if all the traders are trend followers (that is $m = -1$), then*

(1) *for $\delta = 0$, the fundamental steady state is globally asymptotically stable if and only if $0 < \mu < Q/(R-1)$, where $Q = 2a_2(1+q)\sigma_1^2$. In addition, a flip bifurcation occurs along the boundary $\mu = Q/(R-1)$;*

(2) *for $\delta \in (0,1)$, the fundamental steady state is stable for*

$$0 < \mu < \begin{cases} \bar{\mu}_1 & 0 \le \gamma \le \bar{\gamma}_0 \\ \bar{\mu}_2, & \bar{\gamma}_0 \le \gamma, \end{cases}$$

where

$$\bar{\mu}_1 = \frac{Q}{(R-1) - \gamma 2\delta/(1+\delta)}, \quad \bar{\mu}_2 = \frac{(1-\delta)Q}{2\delta[\gamma - (R-1)]}, \quad \bar{\gamma}_0 = (R-1)\frac{(1+\delta)^2}{4\delta}.$$

In addition, a flip bifurcation occurs along the boundary $\mu = \bar{\mu}_1$ for $0 < \gamma \leq \bar{\gamma}_0$ and a Hopf bifurcation occurs along the boundary $\mu = \bar{\mu}_2$ for $\gamma \geq \bar{\gamma}_0$.

Proof. See Appendix A.3. □

The local stability regions and bifurcation boundaries are indicated in Fig. A.2 (a) for $\delta = 0$ and (b) for $\delta \in (0, 1)$ in Appendix A.3, where $\bar{\gamma}_2 = (1 + \delta)(R - 1)/(2\delta)$ is obtained by letting $\bar{\mu}_2 = Q/(R - 1)$. Given that $R = 1 + r/K$ is very close to 1, the value of μ along the flip boundary is very large and $\bar{\gamma}_o$ is close to 0. This implies that, for $\delta = 0$, the fundamental price is stable for a wide range of values of μ. For $\delta \in (0, 1)$, the stability region is mainly bounded by the Hopf bifurcation boundary. Along the Hopf boundary, μ decreases as γ increases, implying that the stability of the steady state is maintained when the speed of the market maker and the extrapolation of the trend followers are balanced. When the fundamental price becomes unstable, the Hopf bifurcation implies that the market price fluctuates (quasi) periodically around the fundamental price. Intuitively, extrapolation of the trend followers results a sluggish reaction of the market price to the fundamental price. The interplay of such sluggish reaction from the trend followers and the stabilizing force from the fundamentalists leads the market price fluctuate around the fundamental price. Numerical simulations indicate that, near the Hopf bifurcation boundary, the price either converges periodically to the fundamental value or oscillates regularly or irregularly. In addition, the Hopf bifurcation boundary shifts to the left when δ increases. This implies that the steady state is stabilizing when more weights are given to the most recent prices.

3.3. **The general case $m \in (-1, 1)$.** We now consider the complete market fraction model DDS with both fundamentalists and trend followers by assuming $m \in (-1, 1)$. Let $a = a_2/a_1$ be the ratio of the absolute risk aversion coefficients. It turns out that the stability and bifurcation of the fundamental steady state are different from the previous two special cases and they are determined jointly by the geometric decay rate and extrapolation rate of the trend followers, the speed of the price adjustment of the fundamentalists towards the fundamental steady state, and the speed of adjustment of the market maker towards the market aggregate demand.

Proposition 3.4. *For DDS (3.1) with $m \in (-1, 1)$,*

(1) *if $\delta = 0$, the fundamental steady state is stable for $0 < \mu < \mu^*$, where*

$$\mu^* = \frac{2Q}{(R-1)(1-m) + a(R+\alpha-1)(1+m)}.$$

In addition, a flip bifurcation occurs along the boundary $\mu = \mu^*$ with $\alpha \in [0, 1]$;

(2) *if $\delta \in (0, 1)$, the fundamental steady state is stable for*

$$0 < \mu < \begin{cases} \mu_1 & 0 \le \gamma \le \gamma_0 \\ \mu_2, & \gamma_0 \le \gamma, \end{cases}$$

where

$$\mu_1 = \frac{1+\delta}{\delta}\frac{Q}{1-m}\frac{1}{\gamma_2 - \gamma}, \qquad \mu_2 = \frac{1-\delta}{\delta}\frac{Q}{1-m}\frac{1}{\gamma - \gamma_1},$$
$$\gamma_1 = (R-1) + a(R+\alpha-1)\frac{1+m}{1-m}, \qquad \gamma_0 = \frac{(1+\delta)^2}{4\delta}\gamma_1, \qquad \gamma_2 = \frac{1+\delta}{2\delta}\gamma_1.$$

In addition, a flip bifurcation occurs along the boundary $\mu = \mu_1$ for $0 < \gamma \le \gamma_0$ and a Hopf bifurcation occurs along the boundary $\mu = \mu_2$ for $\gamma \ge \gamma_0$.

Proof. See Appendix A.3. □

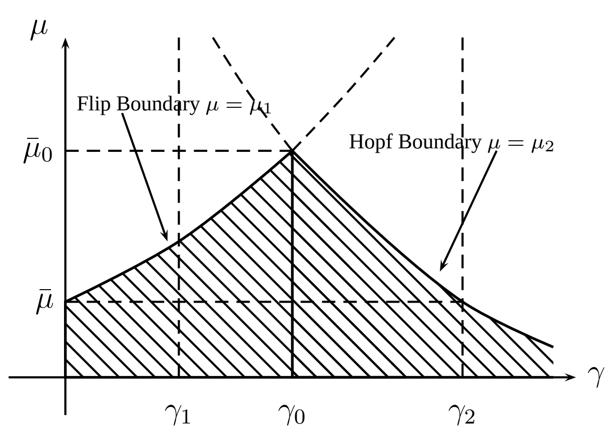


FIGURE 3.1. Stability region and bifurcation boundaries for $m \in (-1, 1)$ and $\delta \in (0, 1)$.

The model with the fundamentalists only can be treated as a degenerated case of the complete model with $\delta = 0$. For $\delta \in (0, 1)$, the fundamental steady state becomes unstable through either flip or Hopf bifurcation, indicated in Fig.3.1, where

$$\bar{\mu}_0 = \frac{2}{1-\delta}\bar{\mu}, \qquad \bar{\mu} = \frac{2Q}{(R-1)(1-m) + a(R+\alpha-1)(1+m)}.$$

Variations of the stability regions and their bifurcation boundaries characterize different impacts of different types of trader on the market price behavior, summarized as follows.

The market fraction has a great impact on the shape of the stability region and its boundaries. It can be verified that γ_1, γ_0, γ_2 and μ_1, μ_2 increase as m increases. This observation has two implications: (i) the local stability region of the parameters (γ, μ) is enlarged as the fraction of the fundamentalists increases and this indicates a stabilizing effect of the fundamentalists; (ii) the flip (Hopf) bifurcation boundary becomes dominant as the fraction of the fundamentalists (trend followers) increases, correspondingly, the market price displays different behavior near the bifurcation boundaries. Numerical simulations of the nonlinear system (3.1) show that the price becomes explosive near the flip bifurcation boundary, but converges to either periodic or quasi-periodic cycles near the Hopf bifurcation boundary.

The speed of price adjustment of the fundamentalists towards the fundamental value has an impact that is negatively correlated to the market fraction. This observation comes from the fact that, as α increases, γ_1 and hence γ_0 and γ_2 decrease. In other words, an increase (decrease) of the fundamentalists fraction is equivalent to a decrease (increase) of the price adjustment speed of the fundamentalists toward the fundamental value.

The memory decay rate of the trend followers has a similar impact on the price behavior as the speed of the price adjustment of the fundamentalists does. This is because, as δ decreases, both γ_0 and γ_2 increase. In particular, as $\delta \to 0$, then $\gamma_0, \gamma_2 \to +\infty$ and the stability and bifurcation is then characterized by the model with the fundamentalists only. On the other hand, as $\delta \to 1$, both γ_0 and γ_2 tend to γ_1 whilst $\bar{\mu}_0$ tends to infinity and the stability and bifurcation are then characterized by the model with the trend followers only. In addition, $\bar{\mu}_o$ increases as δ decreases, implying the steady state is stabilizing as trend followers put more weights on the more recent prices.

The risk aversion coefficients have different impact on the price behavior, depending on the relative risk aversion ratio. Note that $\bar{\mu}$, and hence $\bar{\mu}_0$, increases for $a = a_2/a_1 < a^*$ and decreases for $a > a^*$, where $a^* = (R-1)/(R+\alpha-1) \in (1-1/R, 1]$. Hence the local stability region is enlarged (reduced) when the trend followers are less (more) risk averse than the fundamentalists in the sense of $a_2 < a^* a_1$ ($a_2 > a^* a_1$).

Overall, in terms of the local stability and bifurcation of the fundamental steady state, a similar effect happens for either a high (low) geometric decay rate, or a high (low) market fraction of the trend followers, or a high (low) speed of the price adjustment of the fundamentalists towards the fundamental value. This observation makes us concentrate our statistical analysis of the stochastic model (2.15) on m (the market fraction) and α (the speed of the price adjustment of the fundamentalists toward the fundamental value).

4. Statistical Analysis of the Stochastic Model

In this section, by using numerical simulations, we examine various aspects of the price dynamics of the stochastic heterogeneous asset pricing model (2.15) where both the noisy fundamental price and noisy demand processes are presented. The analysis is conducted by establishing a connection of the price dynamics between SDS (2.15) and its underlying DDS (3.1). In so doing, we are able to obtain some theoretical insights into the generating mechanisms of various statistical properties, including those econometric properties and stylized facts observed in high frequency financial time series.

Our analysis is conducted as follows. As a benchmark, we first briefly review the stylized facts based on the S&P 500. Secondly, we study the connection between the limiting behavior of the stochastic model SDS and the stable attractors of the deterministic shell DDS. This limiting behavior is studied from two different aspects: dynamical behavior and limiting distribution. To study the dynamical behavior, we use the concept of random fixed point to examine the convergence of the market price series in the long-run. The limiting behavior can also be studied by examining the invariant distribution properties from the observed time series. It is found that the asset prices of SDS (2.15) converge to the random fixed point when the DDS (3.1) has either a stable steady state or a stable attractor. When the price of DDS explodes, the price series of SDS does not converge to a random fixed point, but it does converge to an invariant distribution. Thirdly, we use Monte Carlo simulations to conduct a statistical analysis and test on the convergence of the market prices to the fundamental price. It is commonly believed that the market price is mean-reverting to the fundamental price in the long-run, but it can deviate from the fundamental price in the short-run. By using numerical simulation, we analyze market conditions under which this is hold. Finally, by examining the autocorrelation (AC) structure and invariant distribution of (relative) returns near different types of bifurcations, we study the generating mechanism of different AC patterns. Most of our results are very intuitive and can be explained by various behavioral aspects of the model, including the mean reverting of the fundamentalists, the extrapolation of the trend followers, the speed of price adjustment of the market maker, and the market dominance. The statistical analysis and tests are based on Monte Carlo simulations.

4.1. **Financial Time Series and Stylized Facts.** As a benchmark, we include time series plots on prices and returns for the S&P 500 from Aug. 10, 1993 to July 24, 2002 and the corresponding density distributions, autocorrelation coefficients (ACs) of the returns, the absolute returns and the squared returns, and summary statistics of the returns in Appendix B. They share some common stylized facts in high-frequency financial time series, including excess volatility (relative to the dividends and underlying cash flows), volatility clustering (high/low fluctuations are followed by high/low fluctuations), skewness (either negative or positive) and excess kurtosis (compared to the normally distributed returns), long range dependence (insignificant ACs of returns, but significant and decaying ACs for absolute and squared returns), etc. For a comprehensive discussion of stylized facts characterizing financial time series, we refer to Pagan (1996) and Lux (2004).

Recent structural models on asset pricing and heterogeneous beliefs have shown a relatively well understood mechanism of generating volatility clustering, skewness and excess kurtosis. However, these are less clear on the mechanism of generating long-range dependence.[11] In addition, there is a lack of statistical analysis and tests on these mechanisms. Our statistical analysis is based on Monte Carlo simulations, aiming to establish a connection between various AC patterns of the SDS and the bifurcation of the underlying DDS. Such a connection is necessary to understand the mechanism of generating stylized facts, to replicate econometric properties of financial time series, and to calibrate the model to financial data.

In the following discussion, we choose the annual volatility of the fundamental price to be 20% (hence $\sigma_\epsilon = (20/\sqrt{K})\%$ with $K = 250$) and the volatility of the noisy demand $\sigma_\delta = 1$, which is about 1% of the average fundamental price level $\bar{P} = \$100$. For all of the Monte Carlo

[11] See Lux (2004) for a recent survey on possible mechanisms generating long range dependence, including coexistence of multiple attractors and multiplicative noise process.

simulation, we run 1,000 simulations over 6,000 time periods and discard the first 1,000 time periods to wash out possible initial noise effects. Each simulation builds on two independent sets of random numbers, one is for the fundamental price and the other is for the noisy demand. The draws are i.i.d. across the 1,000 simulations, but the same sets of draws are used for different scenarios with different sets of parameters.

4.2. **Random Fixed Point and Limiting Behavior.** One of the primary objectives of this paper is to analyze the limiting behavior of SDS (2.15). For DDS (3.1), the limiting behavior is characterized by either stable fixed points or various stable attractors. For a stochastic dynamic system, the limiting behavior is often characterized by stationarity and invariant probability distributions. We examine invariant distribution properties of SDS when the prices of DDS either converge to a stable attractor (steady state or closed cycle) or explode.

On the other hand, as pointed out in Böhm and Chiarella (2005), the invariance distribution does not provide information about the stability of a stationary solution generated by the stochastic difference system. The theory of random dynamical system (e.g. Arnold (1998)) provides the appropriate concepts and tools to analyze sample paths and investigate their limiting behavior. The central concept is that of a *random fixed point*[12] and its asymptotic stability, which are generalizations of the deterministic fixed point and its stability. Intuitively, a random fixed point corresponds to a stationary solution of a stochastic difference system like (2.15) and the asymptotic stability implies that sample paths converge to the random fixed point wise for all initial conditions of the system. We are interested in the existence and stability of a random fixed point of SDS (2.15) when the deterministic DDS (3.1) displays a stable attractor. Since SDS (2.15) is nonlinear, a general theory on the existence and stability of a random fixed point is not yet available and we conduct our analysis by numerical simulations.

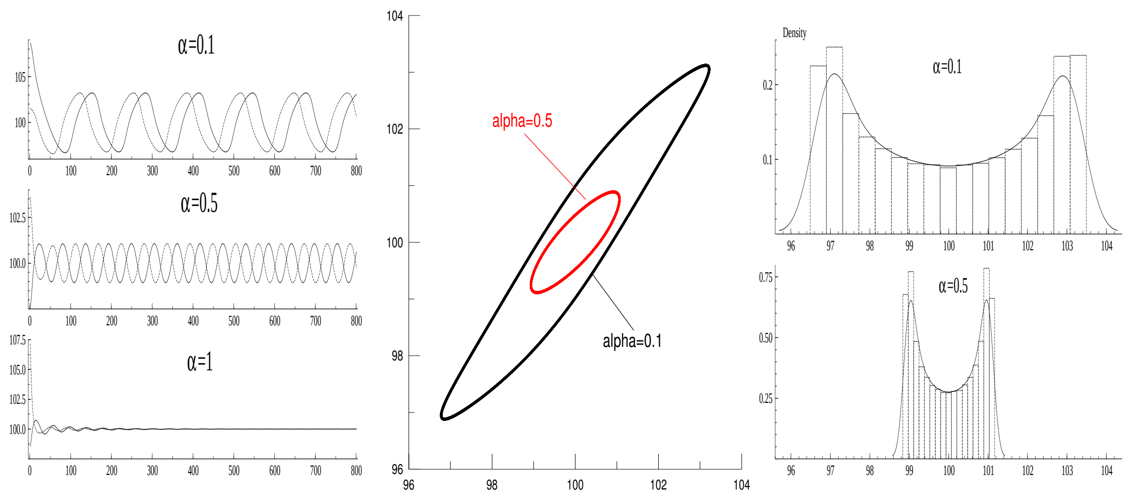
For illustration, we choose the parameters as follows

$$\gamma = 2.1, \quad \delta = 0.85, \quad \mu = 0.2, \quad m = 0, \quad w_{1,0} = 0.5 \quad \text{and} \quad \alpha = 1, 0.5, 0.1, 0. \tag{4.1}$$

Recall that $m = 0$ implies that there are equal numbers of fundamentalists and chartists in the market. For the DDS (3.1) with the set of parameters (4.1), applying Proposition 3.4 implies that

[12]We refer to Arnold (1998) for mathematical definitions of random dynamical systems and of stable random fixed points and Böhm and Chiarella (2005) for economical applications to asset pricing with heterogeneous mean variance preferences.

the fundamental value is locally asymptotically stable for $\alpha = 1$ and unstable for $\alpha = 0.5, 0.1, 0$. Our numerical simulations results for the DDS (3.1) with different values of α are illustrated in Fig. 4.1. Fig. 4.1 (a) shows the time series of prices with different initial values for $\alpha = 0.1, 0.5$ and 1, Fig. 4.1 (b) shows the corresponding limiting phase plots in terms of (P_t, u_t), and Fig. 4.1 (c) shows the limiting probability distributions of the prices for $\alpha = 0.1$ and 0.5 over time period from $t = 1,001$ to $t = 10,000$. For $\alpha = 0$, the prices explode. One can see that, for $\alpha = 1$, the market prices with different initial values converge to the fundamental price. However, for $\alpha = 0.5$ and 0.1, with different initial values, prices do not converge to each other, but converge to the same quasi-periodic cycle (this is demonstrated by the closed orbit in the phase plots). In other words, the prices with different initial values converge to each other in limiting distribution, as indicated by the price probability limiting distributions in Fig. 4.1 (c).



(a) (b) (c)

FIGURE 4.1. Price time series with different initial values for $\alpha = 0.1, 0.5$ and 1 (a) and phase plots of (P_t, u_t) (b) and limiting probability distributions of the prices for $\alpha = 0.1$ and 0.5.

For the parameter set (4.1), Fig.4.2 shows the price dynamics of the corresponding SDS (2.15) with four different values of $\alpha = 1, 0.5, 0.1, 0$ and (arbitrarily) different initial conditions but with a fixed set of noisy fundamental value and demand processes. It is found that, for $\alpha = 1, 0.5$ and 0.1, respectively, there exists a random fixed point and prices with different conditions converge to the fixed random point in the long run. In fact, the convergence only takes about 50, 100 and 400 time periods for $\alpha = 1, 0.5$ and 0.1, respectively. However, there is no such stable random fixed point for $\alpha = 0$ and prices with different initial conditions lead to different random sample paths. In fact, the sample paths are shifted by different initial conditions. This result is very interesting. For $\alpha = 1$, the prices of the DDS with different initial values converge to the stable steady state, while the prices of the SDS with different initial values converge to a random fixed point. For $\alpha = 0.5$ and 0.1, the prices of the DDS with different initial values do not converge to each other, while the prices of the SDS with different initial values converge to a random fixed point.

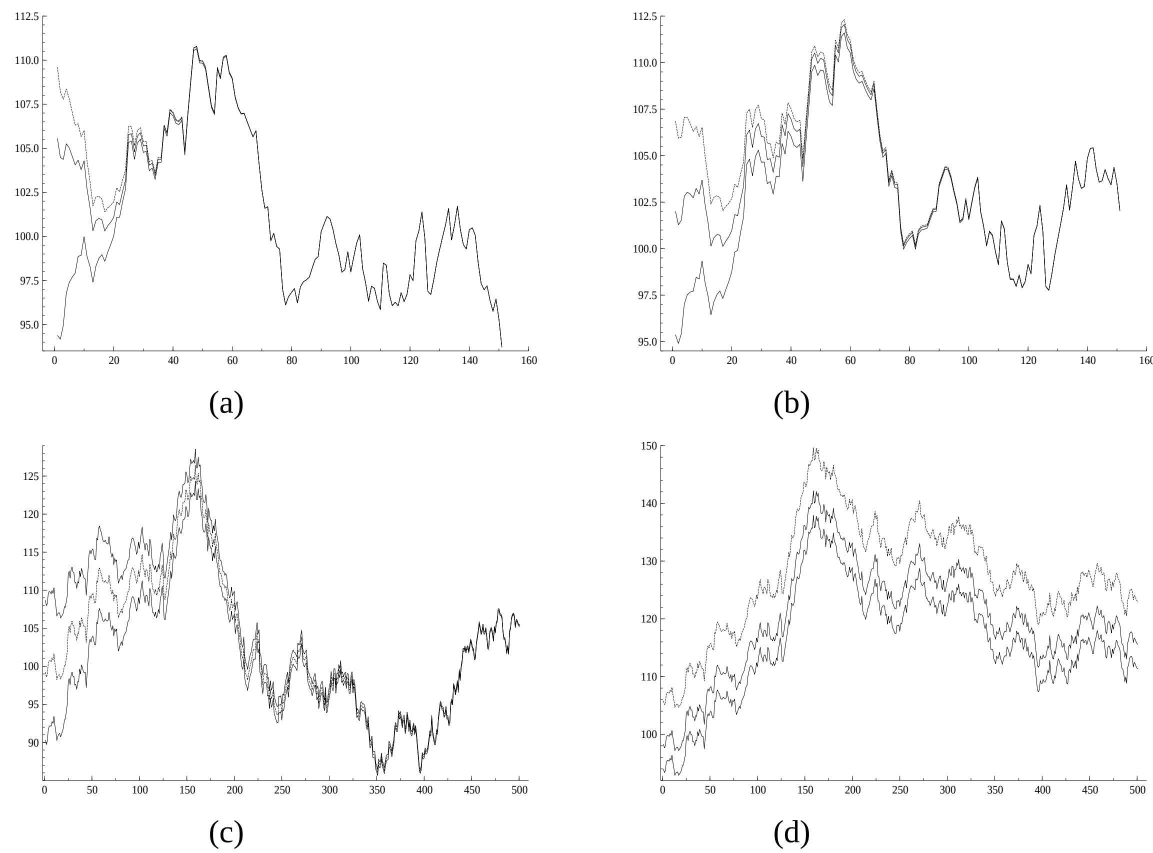


FIGURE 4.2. Price convergence with α=1 (a); 0.5 (b); 0.1 (c); and 0 (d) for different initial conditions.

The long-run behavior can also be characterized by the limiting probability distribution, this is given in Fig. 4.3 for different values of α. In Fig. 4.3 (a), the limiting probability distributions of the market prices and the underlying fundamental price over time period $t = 1,001$ to $t = 10,000$ for $\alpha = 1, 0.9, 0.5, 0.1, 1$ are plotted. The distributions look very similar to the one for the fundamental price for $\alpha = 1, 0.9, 0.5, 0.1$, but different for $\alpha = 0$ (in which the prices of the DDS explode). In Fig. 4.3 (b), we observe a similar feature for the limiting return distributions. However, unlike the price distributions, the return distributions for $\alpha = 1, 0.9, 0.5, 1$ are very different from that for the fundamental price, they all share some non-normality features, including skewness and high kurtosis, as indicated by the results on return statistics and normality tests in Table 4.1. Therefore, we obtain stable invariant distribution (characterized by the

	$\alpha=1$	$\alpha=0.9$	$\alpha=0.5$	$\alpha=0.1$	$\alpha=0$	r_t^*
Mean	-7.64E-06	-9.75E-06	-1.89E-05	-3.38E-05	0.001124	1.60E-07
Median	-8.90E-05	-7.07E-05	-0.000112	-0.000103	-3.01E-06	0.000114
Maximum	0.073622	0.072503	0.070621	0.071766	5.090196	0.045078
Minimum	-0.063119	-0.064302	-0.072816	-0.090166	-4.269424	-0.045625
Std. Dev.	0.013236	0.013129	0.012717	0.012432	0.101814	0.012689
Skewness	0.119060	0.117119	0.095103	0.038494	17.46148	-0.014001
Kurtosis	5.061570	5.098182	5.291521	5.777193	1526.675	2.973831
Jarque-Bera	1794.489	1857.181	2203.019	3216.136	9.68E+08	0.612037
Probability	0.000000	0.000000	0.000000	0.000000	0.000000	0.736373
Sum	-0.076388	-0.097484	-0.189223	-0.338318	11.23968	0.001602
Sum Sq. Dev.	1.751808	1.723556	1.617170	1.545346	103.6512	1.609849

TABLE 4.1. Summary statistics of returns for $\alpha = 1, 0.9, 0.5, 0.1, 0$ and that for the fundamental price.

stable random fixed point) for the SDS when the DDS displays stable attractors. For $\alpha = 0$, the price of the DDS explodes, while the prices of the SDS with different initial values stabilize the price process to different random paths. However, they all converge to the same probability distribution, as indicated in Fig.4.3 (b). This analysis illustrates different characteristics between a stable random fixed point and a stable invariance distribution.

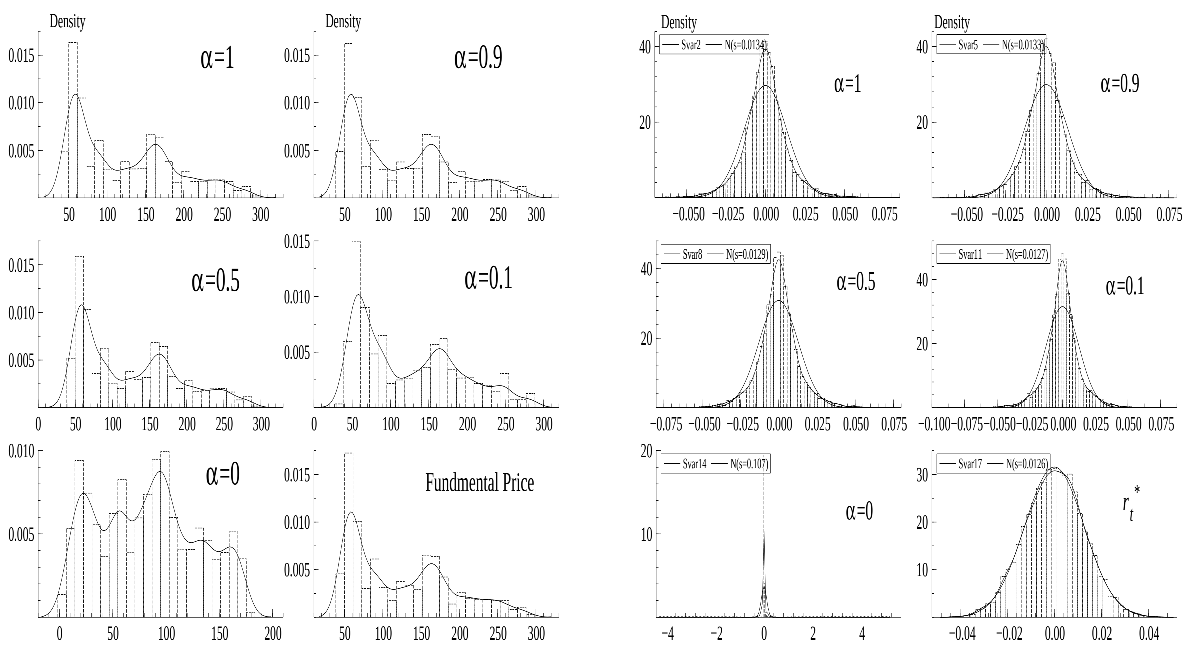


FIGURE 4.3. Limiting probability distributions of prices (a) and returns (b) for $\alpha = 0, 0.1, 0.5$ and 1.

In fact, the above result holds for other selections of parameters. Theoretically, how the stability of the deterministic system and the corresponding stochastic system are related is a difficult problem in general.[13]

4.3. **Convergence of Market Price to the Fundamental Value.** We now turn to the relation between the market price and the fundamental price. It is commonly believed that the market price is mean-reverting to the fundamental price in the long-run, but it can deviate from the fundamental price in the short-run. The following discussion indicates that this is true under certain market conditions.

As we know from the local stability analysis of DDS (3.1) an increase in α has a similar effect as an increase in m. The previous discussion illustrates that, for fixed $m = 0$, as α increases, the speed of convergence of the market price to the random fixed point increases. For SDS (2.15), it is interesting to know how the stable random fixed point is related to the fundamental value process.

To illustrate, for the parameter set (4.1), the averaged time series of the difference of market and fundamental prices $P_t - P_t^*$ based on Monte Carlo simulations are reported in Fig. 4.4. It shows that, as α increases, the deviation of the market price from the fundamental price decreases. That is, as the fundamentalists put more weight on their estimated fundamental price, the deviation of market price from the fundamental price are reduced.

A statistical analysis is conducted by using Monte Carlo simulations for the given set of parameters (4.1) with four different values of α. The resulting Wald statistics to detect the differences between market prices and fundamental prices are reported in Table 4.2. The null hypothesis is specified as, respectively,

- Case 1: $H_0 : P_t = P_t^*, t = 1000, 2000, ..., 5000$;
- Case 2, $H_0 : P_t = P_t^*, t = 3000, 3500, 4000, ..., 5000$;
- Case 3, $H_0 : P_t = P_t^*, t = 4000, 4100, 4200, ..., 5000$;
- Case 4, $H_0 : P_t = P_t^*, t = 4000, 4050, 4100, ..., 5000$;
- Case 5, $H_0 : P_t = P_t^*, t = 4901, 4902, 4903..., 5000$, which refers to the last one hundred periods;

[13] It is well known from the stochastic differential equation literature (e.g. see the examples in Mao (1997), pages 135-141) that, for continuous differential equations, adding noise can have double-edged effect on the stability—it can either stabilize or destabilize the steady state of the differential equations. For our SDS (2.15), numerical simulations show that adding a small (large) noise can stabilizing (destabilize) the price dynamics when parameters are near the flip bifurcation boundary of the DDS (3.1).

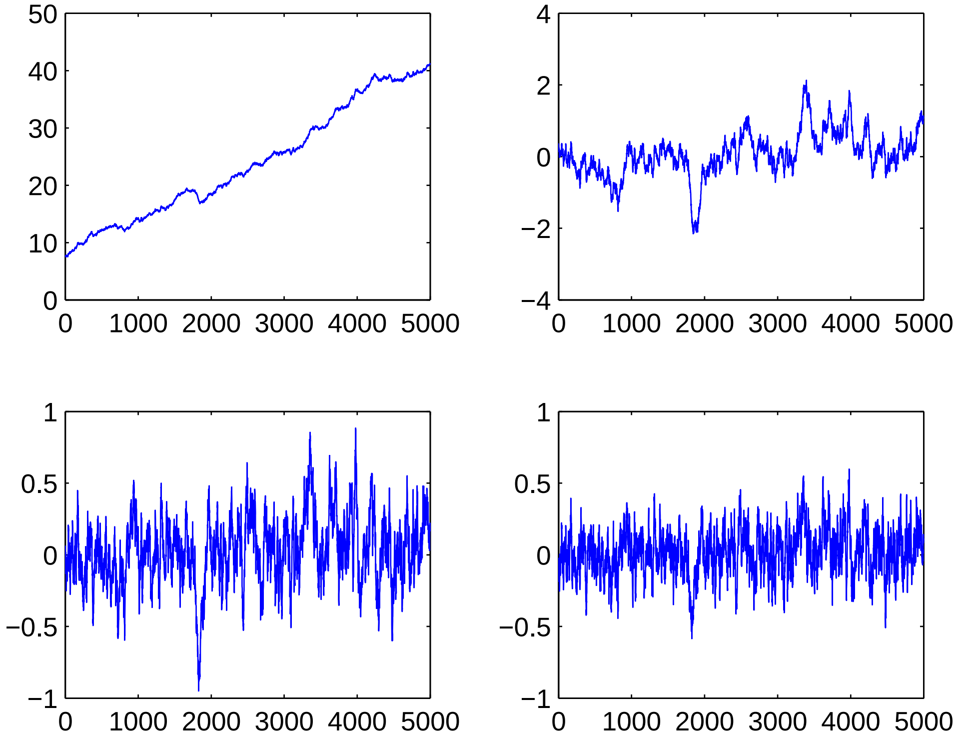


FIGURE 4.4. Time series of price difference $P_t - P_t^*$ with α=0 (top left); 0.1 (top right); 0.5 (second left); and 1 (second right).

- Case 6, $H_0 : P_t = P_t^*, t = 4951, 4952, ..., 5000$, which refers to the last fifty periods.

Notice that the critical values corresponding to the above test statistics come from the χ^2 distribution with degree of freedom 5, 5, 11, 21, 100, and 50, respectively, at the 5% significant level. We see that for $\alpha = 0$, all of the null hypothesis are strongly rejected at the 5% significant level. For $\alpha = 0.5$ and 1, all of the null hypothesis cannot be rejected at the 5% significant level. We also see that when α increases, the resulting Wald statistics decreases (except Case 5 with $\alpha = 1$). This confirms that when α increasing, i.e. when the fundamentalists put more weight on the fundamental price, the differences between the market prices and fundamental prices become smaller.

As we know that an increase in α has similar effect to an increase of the market fraction of the fundamentalists. The above statistic analysis thus implies that, as the fundamentalists dominate the market (as m increases), the market prices follow the fundamental prices closely. Trend extrapolation of the trend followers can drive the market price away from the fundamental price. This result is very intuitive.

	$\alpha = 0$	$\alpha = 0.1$	$\alpha = 0.5$	$\alpha = 1$	Critical value
Case 1	100.585	13.289	5.225	3.698	11.071
Case2	99.817	13.964	6.782	4.358	11.071
Case 3	121.761	24.971	16.041	10.840	19.675
Case 4	148.690	38.038	23.836	19.190	32.671
Case 5	293.963	105.226	99.618	103.299	124.342
Case 6	177.573	50.970	45.043	43.052	67.505

TABLE 4.2. Wald test statistics for the difference between the market price P_t and the fundamental price P_t^* for $n_f = n_c = 0.4$.

4.4. **Bifurcations and Autocorrelation Structure.** Understanding the autocorrelation (AC) structure of returns plays an important role in the market efficiency and predictability. It is often a difficult task to understand the generating mechanism of various AC patterns, in particular those realistic patterns observed in financial time series. It is believed that the underlying deterministic dynamics of the stochastic system play an important role in the AC structure of the stochastic system. But how they are related is not clear. In the following discussion, we try to establish such a connection by analyzing changes of autocorrelation (AC) structures and limiting probability distributions of the stochastic returns when the parameters change near the bifurcation boundaries of the underlying deterministic model. The analysis on the AC structure is conducted through Monte Carlo simulations and the analysis on the limiting distribution is conducted through the probability distribution of returns over time period $t = 1,001$ to $t = 10,000$ for the same underlying noise processes. These analyses lead us to some insights into how particular AC patterns of the stochastic model are characterized by different types of bifurcation of the underlying deterministic system. In doing so, it helps us to understand the mechanism of generating realistic AC patterns.

From our discussion in the previous section, we know that the local stability region of the steady state is bounded by both flip and Hopf bifurcation boundaries in general. To see how the AC structure changes near the different types of the bifurcation boundary, we select two sets of parameters, denoted by (F1) and (H1), respectively,

(F1) $\alpha = 1, \gamma = 0.8, \mu = 5, \delta = 0.85, w_{1,0} = 0.5$ and $m = -0.8, -0.5, -0.3, 0$;

(H1) $\alpha = 1, \gamma = 2.1, \mu = 0.43, \delta = 0.85, w_{1,0} = 0.5$ and $m = -0.95, -0.5, 0, 0.5$.

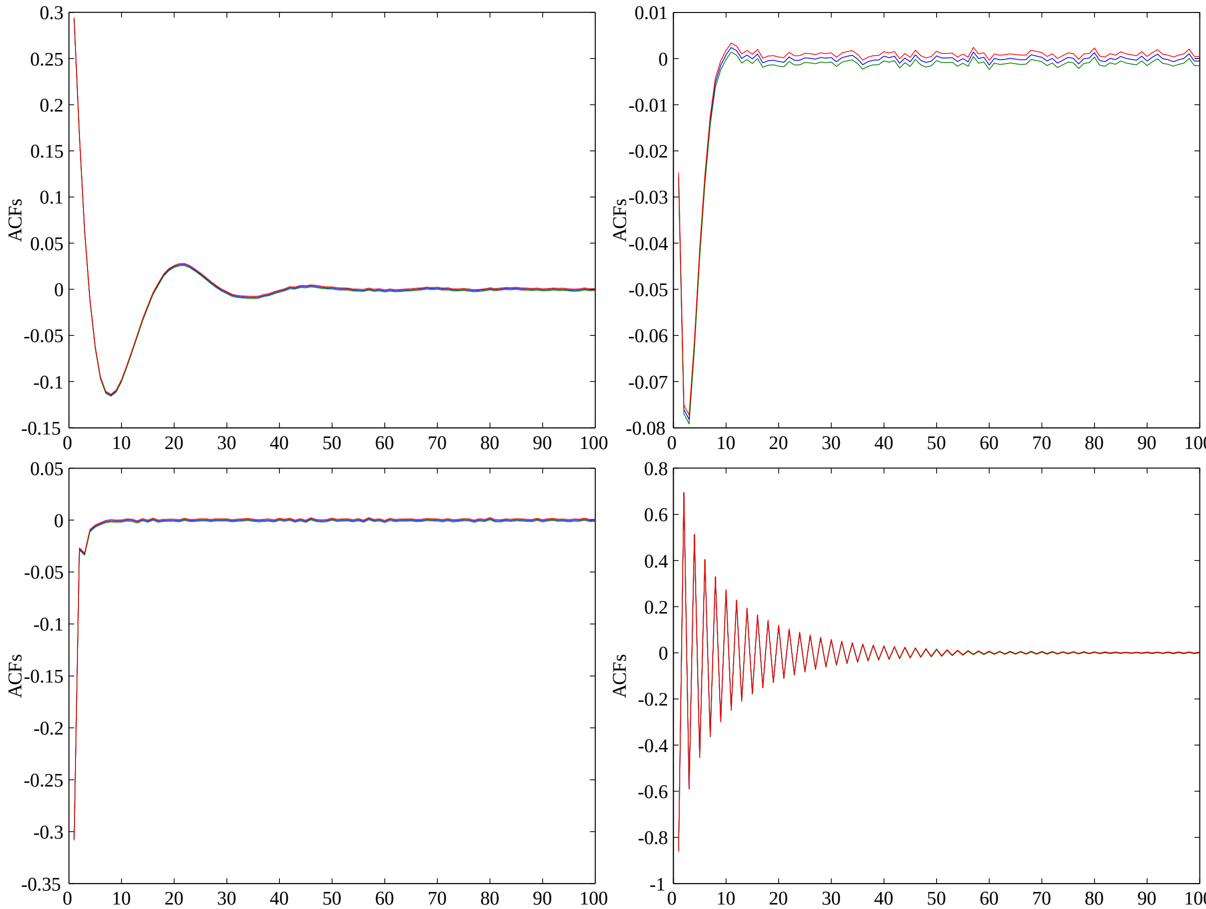


FIGURE 4.5. Monte Carlo simulation on the average ACs of return for $m = -0.8, -0.5, -0.3, 0$ for the parameter set $(F1)$.

For (F1) with different values of m, the steady state of DDS (3.1) is locally stable.[14] However, as m increases, we move closer to the flip boundary.[15] For (H1), there exists a Hopf bifurcation value $\bar{m} \in (0, 0.005)$, the steady state is locally stable for $m = 0.5 \geq \bar{m}$ and unstable for $m = -0.95, -0.5, 0 < \bar{m}$ through a Hopf bifurcation. As m decreases, we are moving close to the Hopf bifurcation boundary initially, and then crossing over the boundary, and then moving away from the boundary. Therefore, an increase in m is stabilizing the steady state. It is interesting to see that the market fraction has different stabilizing effects near different bifurcation boundaries.

For SDS (2.15), Figs. 4.5 and 4.6 report the average ACs of relative return for four different values of m with parameter set (F1) and (H1), respectively. Tables B.2 and B.3 in Appendix B report the average ACs of returns over the first 100 lags, the number in the parentheses are standard errors, the number in the second row for each lag are the total number of ACs that

[14]The solutions become exploded when parameters are near the flip bifurcation boundary and hence we only choose parameters from inside the stable region.

[15]This means that the difference between the given μ and the corresponding flip bifurcation value $\mu_1(m)$ becomes smaller as m increases. It is in this sense that an increase in m is destabilizing the steady state.

are significantly (at 5% level) different from zero among 1,000 simulations. It is found that adding the noise demand does not change the nature of ACs of returns.[16] Given that there is insignificant AC structure from the noisy returns of the fundamental values, the persistent AC patterns displayed in Figs. 4.5-4.6 indicate some connections between AC patterns of SDS (2.15) and the dynamics of the underlying DDS (3.1).

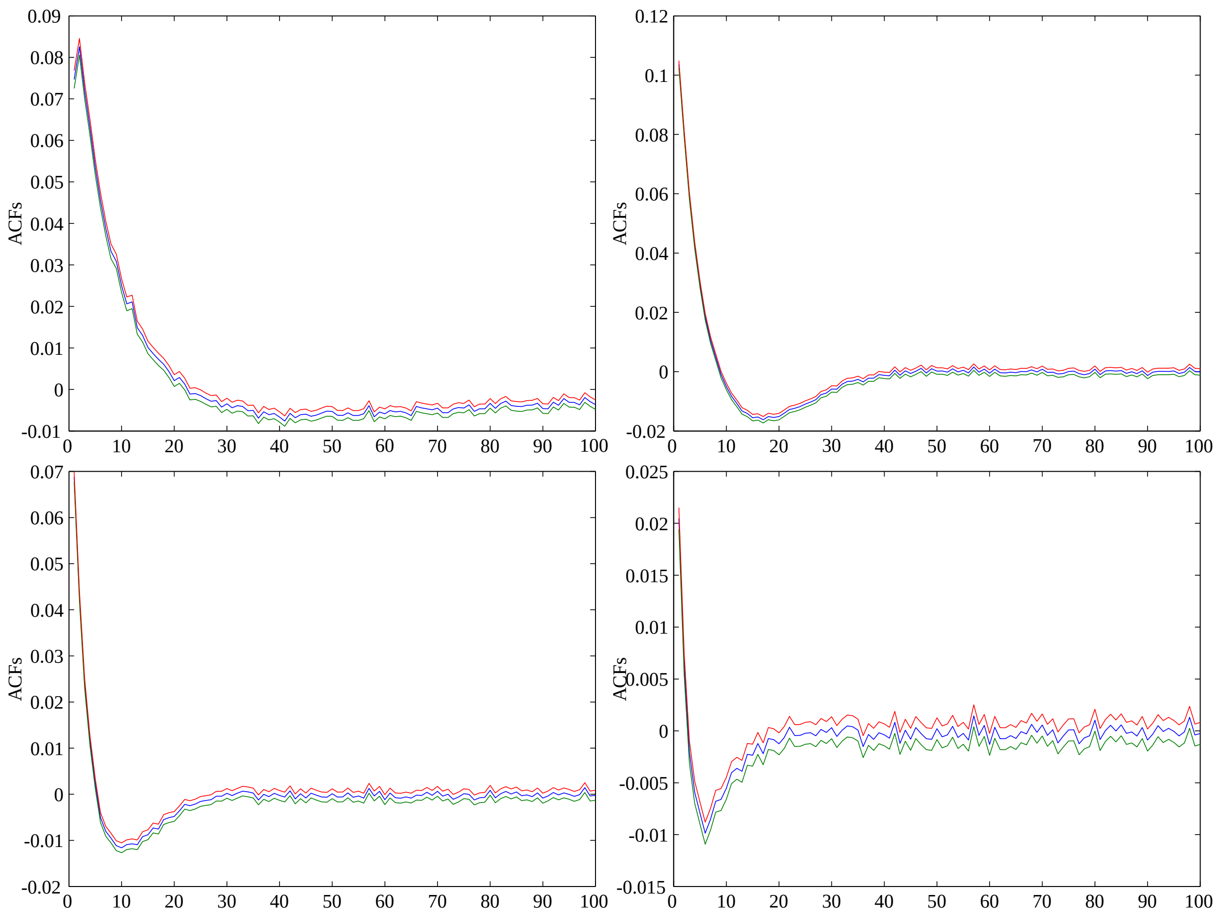


FIGURE 4.6. Monte Carlo simulation on the average ACs of return for $m = -0.95, -0.5, 0, 0.5$ for the parameter set $(H1)$.

For the parameter set (F1), the fundamental value of the underlying DDS (3.1) is locally stable and the AC structure of returns of SDS (2.15) changes as the parameters are moving close to the flip bifurcation boundary. For the deterministic model, we know that an increase of m has a similar effect to an increase of α, the speed of price adjustment of the fundamentalists, or μ, the speed of price adjustment of the market maker. Corresponding to the case of $m = -0.8$ in Fig. 4.5, an **under and over-reaction** pattern characterized by oscillatory decaying ACs with $AC(i) > 0$ for small lags followed by negative ACs for large lags is observed when the

[16]Noisy processes in our model do not change the qualitative nature of the AC of returns, however, they do change the AC patterns of the absolute and squared returns. This issue is addressed in He and Li (2005*b*).

parameters are far away from the flip bifurcation boundary. Intuitively, this results from the constantly price under-adjustment from either the fundamentalists or the market maker. As the parameters are moving toward the flip bifurcation boundary, such as in the case of $m = -0.5, -0.3$ in Fig. 4.5, an **over-reaction** pattern characterized by increasing ACs with $AC(i) < 0$ for small lags i appears. As the parameters move closer to the flip boundary, such as when $m = 0$ in Fig. 4.5, this over-reaction pattern becomes a **strong over-reaction** pattern characterized by an oscillating and decaying ACs which are negative for odd lags and positive for even lags. These results are very intuitive. When the market fractions of the fundamentalists are small, it is effectively equal to a slow price adjustment from either the fundamentalists or market maker, leading to under-reaction. As m increases, such adjustment becomes strong, leading to an over-reaction.[17]

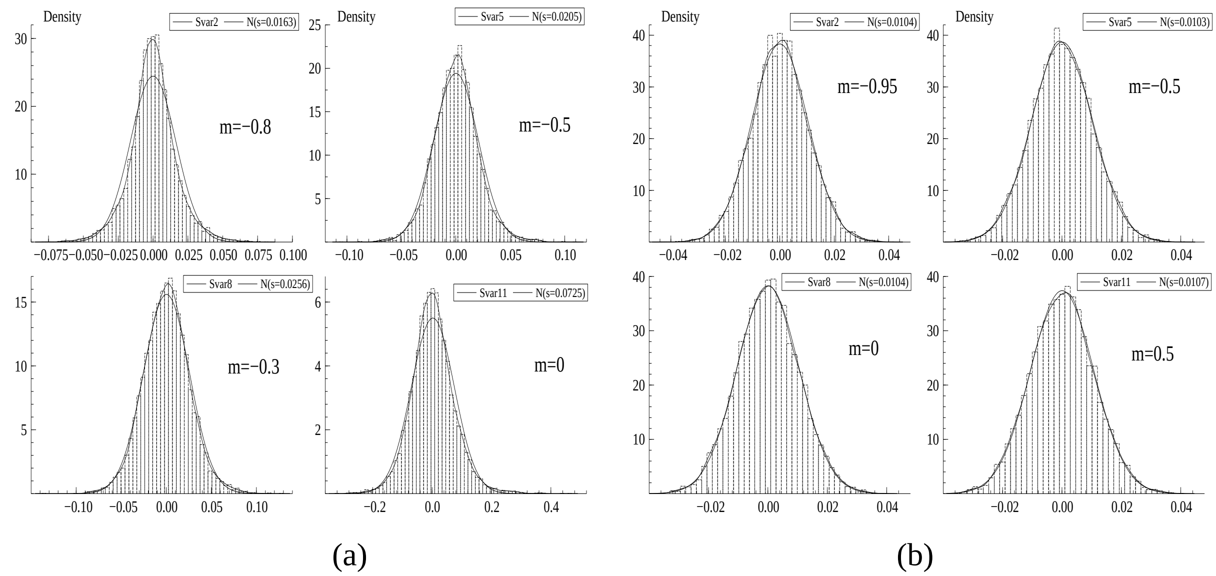


FIGURE 4.7. Limiting probability distributions of market returns for the parameter set (a) $(F1)$ with $m = -0.8, -0.5, -0.3, 0$, and (b) $(H1)$ with $m = -0.95, -0.5, 0, 0.5$.

The limiting distributions of returns and the corresponding statistics near the flip bifurcation boundary for the parameter set $(F1)$ with different values of m are given in Fig. 4.7 (a) and Table 4.3, respectively. It is observed that the returns are not normally distributed with positive skewness and high kurtosis for all values of m. This non-normality underpins the strong AC structure displayed in Fig. 4.5. In addition, as m increases, the standard deviation increases because of the over-reaction of the fundamentalists near the flip bifurcation boundary.

[17] Based on this observation, one can see that both the fundamentalists and market maker need to react to the market price in *a balanced way* in order to generate insignificant AC patterns observed in financial markets. Essentially, this is the mechanism we are using to characterizing the long range dependence in the following subsection.

	$m=-0.8$	$m=-0.5$	$m=-0.3$	$m=0$
Mean	3.95E-05	0.000126	0.000244	5.08E-05
Median	-0.000116	0.000253	0.000336	-1.25E-05
Maximum	0.082283	0.111046	0.125501	0.039912
Minimum	-0.078098	-0.105505	-0.136236	-0.035434
Std. Dev.	0.016142	0.020343	0.025387	0.010419
Skewness	0.072327	0.135512	0.078667	0.039038
Kurtosis	4.547681	4.057518	3.620744	2.997571
Jarque-Bera	1006.767	496.5827	170.8656	2.542365
Probability	0.000000	0.000000	0.000000	0.280500
Sum	0.394987	1.261548	2.438208	0.507550
Sum Sq. Dev.	2.605493	4.137975	6.444583	1.085374

TABLE 4.3. Summary statistics of returns for the parameter set $(F1)$ with $m=-0.8,-0.5,-0.3,0$.

Near the Hopf bifurcation boundary, the AC structure behaves differently when parameters cross the Hopf boundary from the unstable region to the stable region, see Fig. 4.6. For small m, for example $m=-0.95,-0.5$, the steady state of the deterministic model is unstable and it bifurcates to either periodic or quasi-periodic cycles. For the stochastic model, a **strong under-reaction** AC pattern characterized by significantly decaying positive $AC(i)$ for small lags i and insignificantly negative $AC(i)$ for large lags i, as illustrated in Fig. 4.6 for $m=-0.95$.[18] This is partially due to the dominance of the trend followers who follow the lagged learning process. As m increases, for example to $m=-0.5$ and 0, the trend followers becomes less dominated. As the result, the strong under-reaction pattern is replaced by an over-reaction pattern. As m increases further, for example to $m=0.5$, the steady state of the deterministic model becomes stable and the AC structure of the stochastic return in Fig. 4.6 reduces to an insignificant under-reaction pattern.

The limiting distributions of returns and the corresponding statistics near the Hopf bifurcation boundary for the parameter set $(H1)$ for different values of m are given in Fig. 4.7 (b) and Table 4.4, respectively. Different from the previous case near the flip bifurcation boundary, the returns appear to be closer to normal distribution (as indicated by the probabilities of the Jarque-Bera tests) with less significant skewness and kurtosis. This underpins the insignificant AC structure displayed in Fig. 4.6.

[18]The AC structure discussed here are actually combined outcomes of the under-reacting trend followers and over-reacting fundamentalists. This leads price to be under-reacted for short lags, over-reacted for medium lags, and mean-reverted for long lags.

	$m=-0.95$	$m=-0.5$	$m=0$	$m=0.5$
Mean	3.60E-05	4.70E-05	5.08E-05	5.46E-05
Median	6.80E-05	-5.95E-05	-1.25E-05	8.00E-05
Maximum	0.040650	0.041044	0.039912	0.039438
Minimum	-0.042000	-0.035635	-0.035434	-0.034406
Std. Dev.	0.010408	0.010310	0.010419	0.010669
Skewness	0.031815	0.030451	0.039038	0.042038
Kurtosis	3.137758	2.993963	2.997571	2.991432
Jarque-Bera	9.594179	1.560606	2.542365	2.975951
Probability	0.008254	0.458267	0.280500	0.225829
Sum	0.360021	0.469831	0.507550	0.545647
Sum Sq. Dev.	1.083105	1.062926	1.085374	1.138265

TABLE 4.4. Summary statistics of returns for the parameter set $(H1)$ with $m=-0.95,-0.5,0,0.5$.

The above discussion is based on $\alpha = 1$. Similar results are observed for $\alpha < 1$. Fig. B.2 in Appendix B plots the results for the following set of parameters:

$$(FH): \qquad \alpha = 0.5, \gamma = 0.8, \mu = 5, \delta = 0.85, \qquad m = -0.9, -0.5, 0, 0.9.$$

In this case, small values of m are close to the Hopf boundary and large values of m are close to the flip boundary. As we can see from the AC patterns in Fig. B.2 in Appendix B that, as m increases, the AC patterns change from strong under-reaction to under- and over-reaction, and to over-reaction, and then to strong over-reaction.

In all cases, the ACs decay and become insignificant after the first few lags (the first 5 lags for under/over-reaction and the first 10 lags for strong reaction). Briefly, activity of the fundamentalists (either high fraction or high speed of price adjustment) are responsible for over-reaction AC patterns and extrapolation from the trend followers are responsible for the under-reaction AC patterns. In addition, a strong under-reaction AC patterns of SDS is in general associated with Hopf bifurcation of the DDS, a strong over-reaction AC pattern is associated with flip bifurcation, and under and over-reaction AC patterns are associated with both types of bifurcation (depending on their dominance). This statistical analysis on both the AC structure and limiting distribution gives us insights into how the AC structure of the SDS are affected by different types of bifurcation of the underlying DDS.

4.5. **Some other issues.** One of the related issues to our early discussion is the long-range dependence founded in daily financial time series including the S&P 500. It corresponds to an insignificant AC patterns for the returns, but significant AC patterns for the absolute returns and squared returns. Guided by the above analysis, we select following set of parameters: $\alpha = 0.1, \gamma = 0.3, \mu = 2, m = 0, \delta = 0.85, b = 1$. For this set of parameter, the steady state fundamental price $\bar{P}$ of the DDS is locally asymptotically stable. The price and return behaviors are reported in Fig. 4.8.

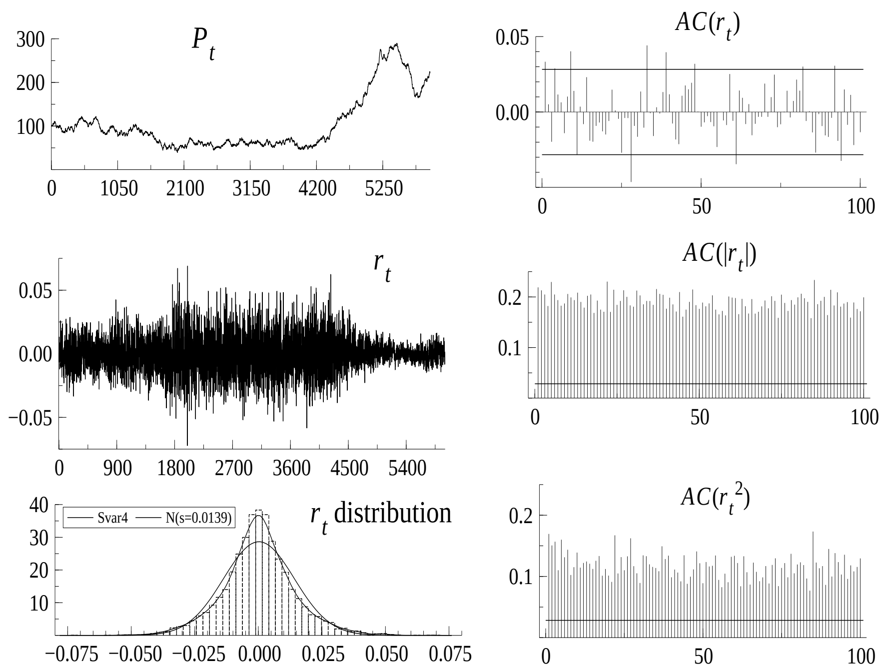


FIGURE 4.8. Time series on prices and returns, density distribution and autocorrelation coefficients (ACs) of the returns, the squared returns and the absolute returns.

In this case, we observe from Figure 4.8 a relatively high kurtosis, volatility clusterings, insignificant ACs for returns, but significant ACs for the absolute and squared returns. This result shows that the model is able to produce relatively realistic volatility clustering and the long-range dependence. A more detailed analysis of the generating mechanism on the long-range dependence and statistical estimates and tests based on Monte Carlo simulation can be found in He and Li (2005*b*).

Another related issue is the profitability and survivability of the fundamentalists and chartists. A systematic analysis of how different, fixed fractions affect survivability and profitability under the current framework is examined in He and Li (2005*a*). Such an approach is perhaps less general than the strategy switching models (e.g. Brock and Hommes (1998)) in which the market fractions are endogenous. We leave this to the future study.

5. Conclusion

It is interesting and important to see how the deterministic dynamics and noise interact with each other. A theoretical understanding of the connections between certain time series properties of the stochastic system and its underlying deterministic dynamics is important but difficult, and a statistical analysis based on various econometric tools seems necessary. Such an analysis helps us to understand potential sources of generating realistic time series properties.

The model proposed in this paper introduces a market fractions model with heterogeneous traders in a simple asset-pricing framework. It contributes to the literature by incorporating a realistic trading period, which eliminates the untenable risk-free rate assumption. By focusing on different aspects of financial market behavior including market dominance and under and over-reaction, we investigate the relationship between deterministic forces and stochastic elements of the stochastic model. A statistical analysis based on Monte Carlo simulations shows that the limiting behavior and convergence of the market prices can be characterized by the stability and bifurcation of the underlying deterministic system. In particular, we show that various under and over-reaction autocorrelation patterns of returns can be characterized by the bifurcation nature of the deterministic system. The model is able to generate some stylized facts, including skewness, high kurtosis, volatility clustering and long-range dependence, observed in high-frequency financial time series.

It is worth emphasizing that all these interesting qualitative and quantitative features arise from our simple market fraction model with fixed market fraction. It would be interesting to extend our analysis from the current model to a changing fraction model developed recently in Dieci *et al.* (2006), in which some part of the market fractions are governed by the herding mechanism (for instance, see Lux and Marchesi (1999)) and the other part follows some evolutionary adaptive processes (see Brock and Hommes (1997,1998) for instance). Taking together the herding and switching mechanisms and the findings in this paper, we hope we can better understand and characterize a large part of the stylized facts of financial data. We hope this will lead to better models for calibrations.

APPENDIX A. PROOFS OF PROPOSITIONS

A.1. **Proof of Proposition 3.1.** For $P_t^* = \bar{P}$, the demand function for the fundamentalists becomes

$$z_{1,t} = \frac{(1-\alpha-R)(P_t-\bar{P})}{a_1(1+r^2)\sigma_1^2}.$$

Let $(P_t, u_t, v_t) = (P_0, u_0, v_0)$ be the steady state of the system. Then (P_0, u_0, v_0) satisfies

$$P_0 = P_0 + \frac{\mu}{2}\Big[(1+m)\frac{(1-\alpha-R)(P_0-\bar{P})}{a_1(1+r^2)\sigma_1^2} + (1-m)\frac{\gamma(P_0-u_0)-(R-1)(P_0-\bar{P})}{a_2\sigma_1^2(1+r^2+b\,v_0)}\Big], \tag{A.1}$$

$$u_0 = \delta u_0 + (1-\delta)P_0, \tag{A.2}$$

$$v_0 = \delta v_0 + \delta(1-\delta)(P_0-u_0)^2. \tag{A.3}$$

One can verify that $(P_0, u_0, v_0) = (\bar{P}, \bar{P}, 0)$ satisfies (A.1)-(A.3); that is the fundamental steady state is one of the steady state of the system (3.1). It follows from (A.2)-(A.3) and $\delta \in [0,1)$ that $P_0 = u_0, v_0 = 0$. This together with (A.1) implies that $P_0 = \bar{P}$. In fact, if $P_0 \neq \bar{P}$, then (A.1) implies that

$$\frac{1+m}{a_1}(1-\alpha-R) + \frac{1-m}{a_2}(1-R) = 0. \tag{A.4}$$

However, since $\alpha \in [0,1]$, $R = 1+r/K > 1$ and $m \in [-1,1]$, equation (A.4) cannot be hold. Therefore the fundamental steady state is the unique steady state of the system.

A.2. **Proof of Proposition 3.2.** For $P_t^* = \bar{P}$ and $m = 1$, equation (3.1) becomes

$$P_{t+1} = P_t - \mu\frac{(R+\alpha-1)(P_t-\bar{P})}{a_1(1+r^2)\sigma_1^2}, \tag{A.5}$$

which can be rewritten as

$$P_{t+1} - \bar{P} = \lambda[P_t - \bar{P}], \tag{A.6}$$

where

$$\lambda \equiv 1 - \mu\frac{R+\alpha-1}{a_1(1+r^2)\sigma_1^2}.$$

Obviously, from (A.6), the fundamental price $\bar{P}$ is globally asymptotically attractive if and only if $|\lambda| < 1$, which in turn is equivalent to $0 < \mu < \mu_o$.

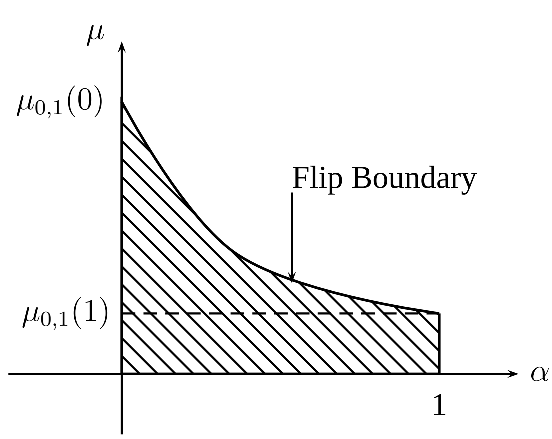


FIGURE A.1. Stability region and bifurcation boundary for $m = 1$.

A.3. **Proof of Propositions 3.3 and 3.4.** For $P_t^* = \bar{P}$, system (3.1) is reduced to the following 3-dimensional difference deterministic system

$$\begin{cases} P_{t+1} = F_1(P_t, u_t, v_t), \\ u_{t+1} = F_2(P_t, u_t, v_t), \\ v_{t+1} = F_3(P_t, u_t, v_t), \end{cases} \tag{A.7}$$

where

$$\begin{aligned} F_1(P,u,v) &= P + \frac{\mu}{2}\Big[(1+m)\frac{(1-\alpha-R)(P-\bar{P})}{a_1(1+r^2)\sigma_1^2} \\ &\qquad + (1-m)\frac{\gamma(P-u)-(R-1)(P-\bar{P})}{a_2\sigma_1^2(1+r^2+b\,v)}\Big], \\ F_2(P,u,v) &= \delta u + (1-\delta)F_1(P,u,v), \\ F_3(P,u,v) &= \delta v + \delta(1-\delta)(F_1-u)^2. \end{aligned}$$

Denote

$$a = \frac{a_2}{a_1}, \qquad Q = 2a_2(1+r^2)\sigma_1^2.$$

At the fundamental steady state $(\bar{P}, \bar{P}, 0)$,

$$\begin{aligned} &\frac{\partial F_1}{\partial P} = A \equiv 1 + \frac{\mu}{Q}[(1+m)a(1-\alpha-R) + (1-m)(1+\gamma-R)], \\ &\frac{\partial F_1}{\partial u} = B \equiv -\frac{\mu\gamma(1-m)}{Q}, \qquad \frac{\partial F_1}{\partial v} = 0; \\ &\frac{\partial F_2}{\partial P} = (1-\delta)A, \qquad \frac{\partial F_2}{\partial u} = C \equiv \delta + (1-\delta)B, \qquad \frac{\partial F_2}{\partial v} = 0; \\ &\frac{\partial F_3}{\partial P} = \frac{\partial F_3}{\partial u} = \frac{\partial F_3}{\partial v} = 0. \end{aligned}$$

Then the Jacobian matrix of the system at the fundamental steady state J is given by

$$J = \begin{pmatrix} A & B & 0 \\ (1-\delta)A & C & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{A.8}$$

and hence the corresponding characteristic equation becomes

$$\lambda\Gamma(\lambda) = 0,$$

where

$$\Gamma(\lambda) = \lambda^2 - [A + \delta + (1-\delta)B]\lambda + \delta A.$$

It is well known that the fundamental steady state is stable if all three eigenvalues λ_i satisfy $|\lambda_i| < 1$ $(i = 1,2,3)$, where $\lambda_3 = 0$ and $\lambda_{1,2}$ solve the equation $\Gamma(\lambda) = 0$.

For $\delta = 0$, $\Gamma(\lambda) = \lambda[\lambda - (A+B)]$. The first result of Proposition 3.3 is then follows from $-1 < \lambda = A + B < 1$ and $\lambda = -1$ when $A + B = 1$.

For $\delta \in (0,1)$, the fundamental steady state is stable if

(i). $\Gamma(1) > 0$;
(ii). $\Gamma(-1) > 0$;
(iii). $\delta A < 1$.

It can be verified that

(i). For $\alpha \in [0,1]$, $\Gamma(1) > 0$ holds;
(ii). $\Gamma(-1) > 0$ is equivalent to

$$\text{either} \qquad \gamma \geq \gamma_2 \qquad \text{or} \qquad 0 < \gamma < \gamma_2 \qquad \text{and} \qquad 0 < \mu < \mu_1,$$

where

$$\gamma_2 = \frac{1+\delta}{2\delta}[(R-1) + a(R+\alpha-1)\frac{1+m}{1-m}],$$
$$\mu_1 = \frac{1+\delta}{\delta}\frac{Q}{1-m}\frac{1}{\gamma_2-\gamma}.$$

(iii). The condition $\delta A < 1$ is equivalent to

$$\text{either} \quad \gamma \le \gamma_1 \quad \text{or} \quad \gamma > \gamma_1 \quad \text{and} \quad 0 < \mu < \mu_2,$$

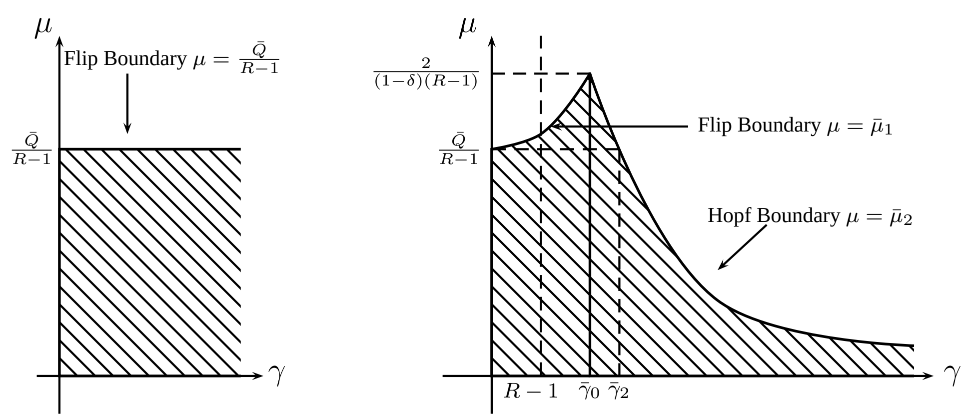
where

$$\gamma_1 = (R-1) + a(R+\alpha-1)\frac{1+m}{1-m},$$
$$\mu_2 = \frac{1-\delta}{\delta}\frac{Q}{1-m}\frac{1}{\gamma-\gamma_1}.$$

Noting that, for $\delta \in (0,1)$, $\gamma_1 < \gamma_0 < \gamma_2$, where

$$\gamma_0 = \frac{(1+\delta)^2}{4\delta}\left[(R-1) + a(R+\alpha-1)\frac{1+m}{1-m}\right]$$

solves the equation $\mu_1 = \mu_2$. Also, μ_1 is an increasing function of γ for $\gamma < \gamma_2$ while μ_2 is a decreasing function of γ for $\gamma > \gamma_1$. Hence the two conditions for the stability are reduced to $0 < \mu < \mu_1$ for $0 \le \gamma \le \gamma_0$ and $0 \le \mu \le \mu_2$ for $\gamma > \gamma_0$. In addition, the two eigenvalues of $\Gamma(\lambda) = 0$ satisfy $\lambda_1 = -1$ and $\lambda_2 \in (-1,1)$ when $\mu = \mu_1$ and $\lambda_{1,2}$ are complex numbers satisfying $|\lambda_{1,2}| < 1$ when $\mu = \mu_2$. Therefore, a flip bifurcation occurs along the boundary $\mu = \mu_1$ for $0 < \gamma \le \gamma_0$ and a Hopf bifurcation occurs along the boundary $\mu = \mu_2$ for $\gamma \ge \gamma_0$.



(a) $\delta = 0$ (b) $\delta \in (0,1)$

FIGURE A.2. Stability region and bifurcation boundaries for the trend followers and market maker model with $\delta = 0$ (a) and $\delta \in (0,1)$ (b).

APPENDIX B. MONTE CARLO SIMULATIONS AND STATISTICAL RESULTS

Econometric Properties and Statistics of the S&P 500. In this appendix, we include time series plots on prices and returns for the S&P 500 from Aug. 10, 1993 to July 24, 2002 in Fig.B.1. The corresponding density distributions, autocorrelation coefficients (ACs) of returns, absolute returns and squared returns are also illustrated in Fig. B.1. Table B.1 presents summary statistics of the returns.

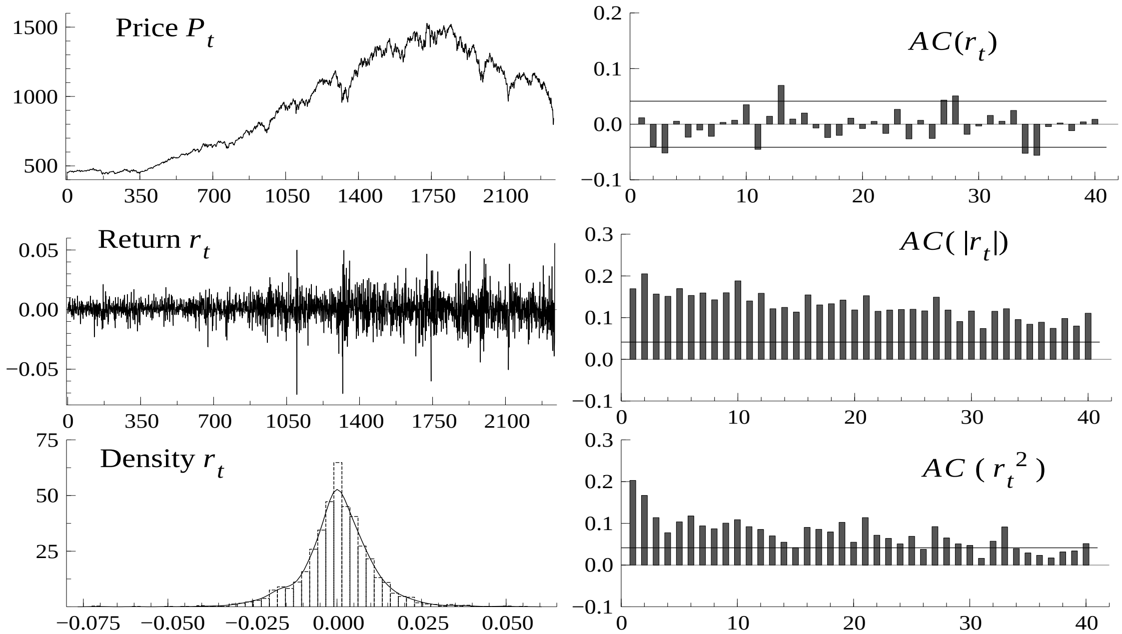


FIGURE B.1. Time series on prices and returns, density distribution and autocorrelation coefficients (ACs) of the returns, the squared returns and the absolute returns for the S&P 500 from Aug. 10, 1993 to July 24, 2002.

TABLE B.1. Summary statistics of returns for the S&P 500.

Index	Mean	Median	Max.	Min.	Std. Dev.	Skew.	Kurt.	Jarque-Bera
S&P500	0.000194	0.0000433	0.057361	-0.070024	0.0083	-0.504638	8.215453	2746.706

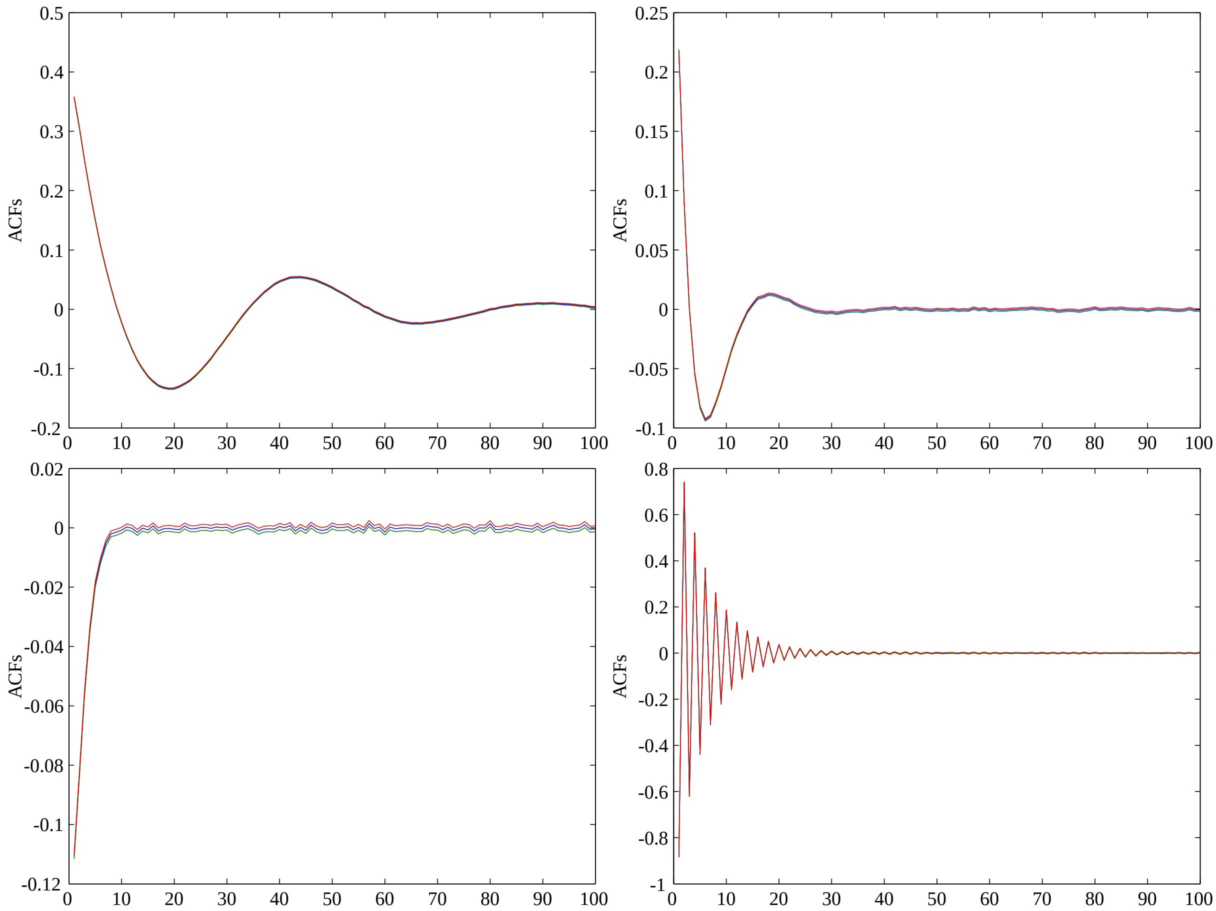
TABLE B.2. Autocorrelations of r_t for the flip-set parameter $(F1)$.

Lag	$m=-0.8$	$m=-0.5$	$m=-0.3$	$m=0$
1	0.2933 (0.0169) 993	-0.0256 (0.0149) 455	-0.3076 (0.0136) 1000	-0.8602 (0.0084) 1000
2	0.1664 (0.0162) 988	-0.0760 (0.0152) 935	-0.0278 (0.0169) 720	0.6939 (0.0161) 1000
3	0.0636 (0.0161) 883	-0.0782 (0.0157) 915	-0.0328 (0.0168) 456	-0.5899 (0.0205) 1000
4	-0.0112 (0.0164) 297	-0.0621 (0.0158) 826	-0.0102 (0.0168) 115	0.5123 (0.0233) 998
5	-0.0630 (0.0168) 868	-0.0420 (0.0158) 625	-0.0058 (0.0167) 79	-0.4528 (0.0250) 986
6	-0.0958 (0.0168) 949	-0.0262 (0.0158) 379	-0.0034 (0.0167) 70	0.4033 (0.0262) 978
7	-0.1116 (0.0169) 968	-0.0134 (0.0158) 163	-0.0014 (0.0167) 72	-0.3631 (0.0269) 969
8	-0.1148 (0.0169) 976	-0.0052 (0.0158) 57	-0.0006 (0.0166) 54	0.3282 (0.0274) 955
9	-0.1102 (0.0169) 966	-0.0015 (0.0159) 58	-0.0010 (0.0167) 53	-0.2981 (0.0278) 934
10	-0.0989 (0.0169) 953	0.0008 (0.0159) 63	-0.0009 (0.0167) 57	0.2712 (0.0280) 916
20	0.0248 (0.0179) 338	-0.0006 (0.0160) 51	-0.0001 (0.0167) 57	0.1188 (0.0278) 690
30	-0.0036 (0.0181) 96	0.0002 (0.0160) 51	0.0002 (0.0167) 54	0.0565 (0.0268) 463
40	-0.0020 (0.0180) 88	0.0005 (0.0160) 39	0.0007 (0.0167) 47	0.0291 (0.0262) 299
50	0.0015 (0.0180) 77	0.0006 (0.0160) 66	0.0009 (0.0167) 56	0.0150 (0.0259) 230
60	-0.0017 (0.0181) 99	-0.0014 (0.0161) 56	-0.0013 (0.0167) 54	0.0059 (0.0259) 218
70	0.0012 (0.0181) 84	0.0003 (0.0161) 54	0.0001 (0.0167) 50	0.0046 (0.0259) 197
80	0.0005 (0.0180) 74	0.0013 (0.0161) 76	0.0014 (0.0167) 64	0.0032 (0.0258) 181
90	-0.0006 (0.0181) 84	-0.0006 (0.0161) 64	-0.0007 (0.0167) 54	0.0016 (0.0259) 184
100	-0.0003 (0.0181) 69	-0.0005 (0.0162) 48	-0.0001 (0.0168) 52	0.0023 (0.0258) 192

TABLE B.3. Autocorrelations of r_t for the Hopf-set parameter $(H1)$.

Lag	$m=-0.95$	$m=-0.5$	$m=0$	$m=0.5$
1	0.0746 (0.0345) 898	0.1037 (0.0196) 964	0.0688 (0.0176) 582	0.0205 (0.0168) 730
2	0.0825 (0.0326) 811	0.0802 (0.0189) 868	0.0429 (0.0174) 469	0.0064 (0.0169) 687
3	0.0720 (0.0315) 788	0.0593 (0.0187) 672	0.0241 (0.0173) 434	-0.0020 (0.0170) 618
4	0.0631 (0.0309) 756	0.0426 (0.0183) 493	0.0116 (0.0173) 422	-0.0059 (0.0171) 529
5	0.0535 (0.0301) 721	0.0294 (0.0182) 380	0.0023 (0.0174) 436	-0.0079 (0.0171) 418
6	0.0456 (0.0292) 677	0.0185 (0.0182) 301	-0.0050 (0.0173) 398	-0.0099 (0.0171) 339
7	0.0388 (0.0288) 587	0.0107 (0.0180) 272	-0.0080 (0.0173) 366	-0.0085 (0.0170) 244
8	0.0333 (0.0287) 498	0.0049 (0.0179) 257	-0.0095 (0.0171) 325	-0.0068 (0.0170) 161
9	0.0309 (0.0278) 433	-0.0009 (0.0178) 290	-0.0111 (0.0173) 313	-0.0066 (0.0170) 154
10	0.0250 (0.0268) 358	-0.0050 (0.0177) 281	-0.0116 (0.0172) 245	-0.0055 (0.0170) 106
20	0.0021 (0.0230) 88	-0.0152 (0.0175) 228	-0.0048 (0.0171) 62	-0.0012 (0.0170) 53
30	-0.0035 (0.0215) 78	-0.0058 (0.0174) 76	0.0002 (0.0171) 53	0.0003 (0.0170) 58
40	-0.0066 (0.0201) 84	-0.0013 (0.0175) 54	-0.0003 (0.0172) 50	-0.0004 (0.0170) 47
50	-0.0053 (0.0191) 80	0.0002 (0.0177) 56	0.0001 (0.0172) 63	0.0002 (0.0170) 62
60	-0.0059 (0.0193) 85	-0.0005 (0.0175) 53	-0.0012 (0.0172) 60	-0.0013 (0.0171) 54
70	-0.0045 (0.0190) 72	0.0008 (0.0175) 61	0.0006 (0.0172) 59	0.0006 (0.0171) 56
80	-0.0034 (0.0186) 73	0.0008 (0.0175) 61	0.0009 (0.0172) 61	0.0010 (0.0170) 58
90	-0.0046 (0.0185) 73	-0.0013 (0.0176) 60	-0.0008 (0.0172) 65	-0.0009 (0.0171) 63
100	-0.0037 (0.0183) 56	-0.0001 (0.0178) 55	-0.0002 (0.0173) 50	-0.0003 (0.0171) 43

FIGURE B.2. Monte Carlo simulation on the average ACs of return for $m = -0.9$ (top left), -0.5 (top right), 0 (bottom left), 0.9 (bottom right) for the parameter set (FH).



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