Lecture Two

Market Behaviour and Statistic Properties of a Market Fraction Model of Heterogeneous Agents

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Financial Market Behaviour with Heterogeneous Investors
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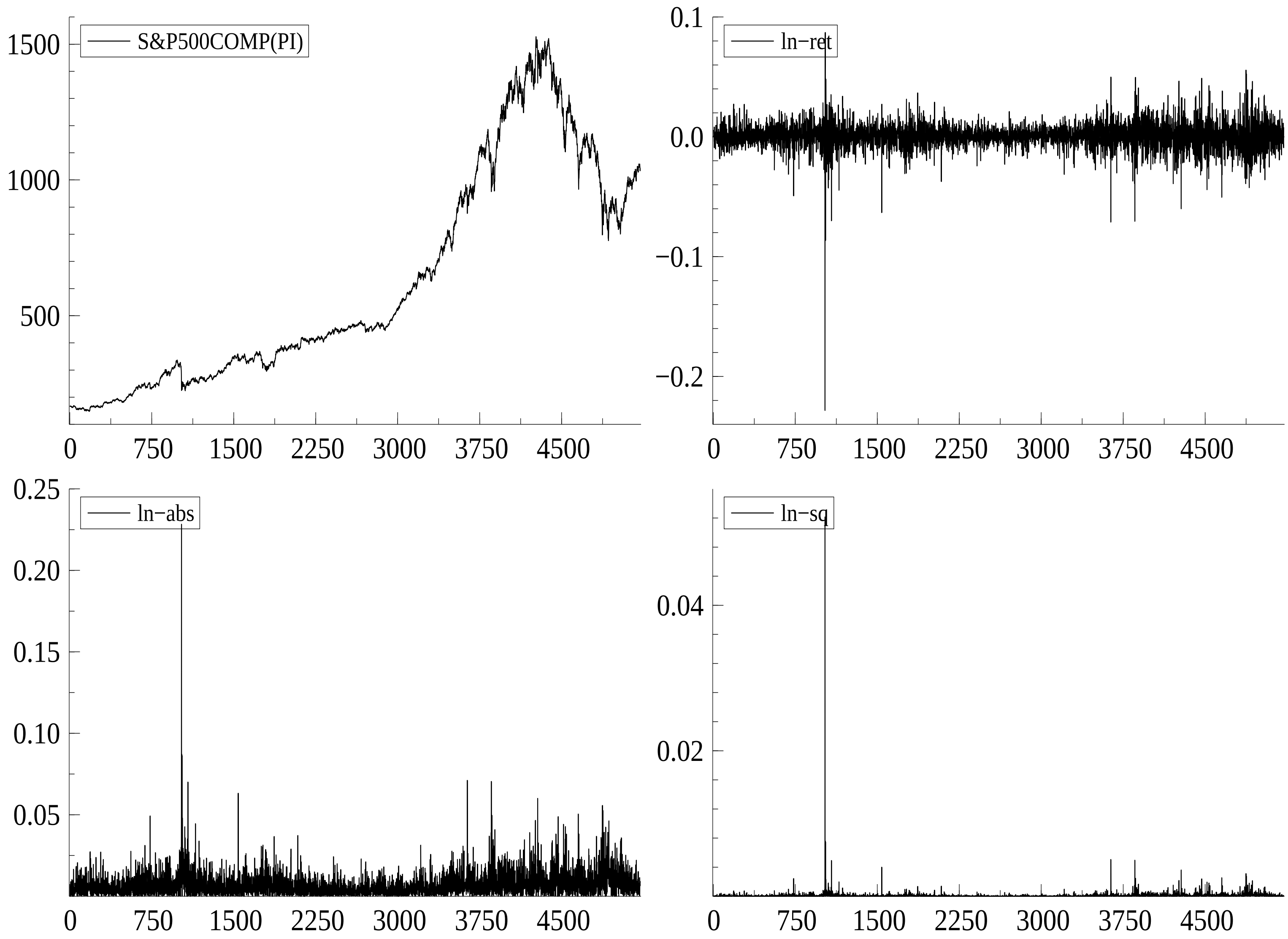
1 Literature Review and Motivation

1.1 Stylized Factors and Long Memory

- **Homogeneous Expectations and Representative Agents**
- **Efficient Market Hypothesis**
- **Excess volatility—relative to the dividends and underlying cash flows**
- **Volatility clustering—high/low fluctuations are followed by high/low fluctuations**
- **Skewness and higher excess kurtosis**
- **Long memory—the long-range dependence—hyperbolic decline of its autocorrelation function (Ding Engle and Granger (1993))**


S&P500COMP(PI)
1500
1000
500
0
750
1500
2250
3000
3750
4500
0.1
ln−ret
0.0
−0.1
−0.2
0.25
ln−abs
0.20
0.15
0.10
0.05
ln−sq
0.04
0.02

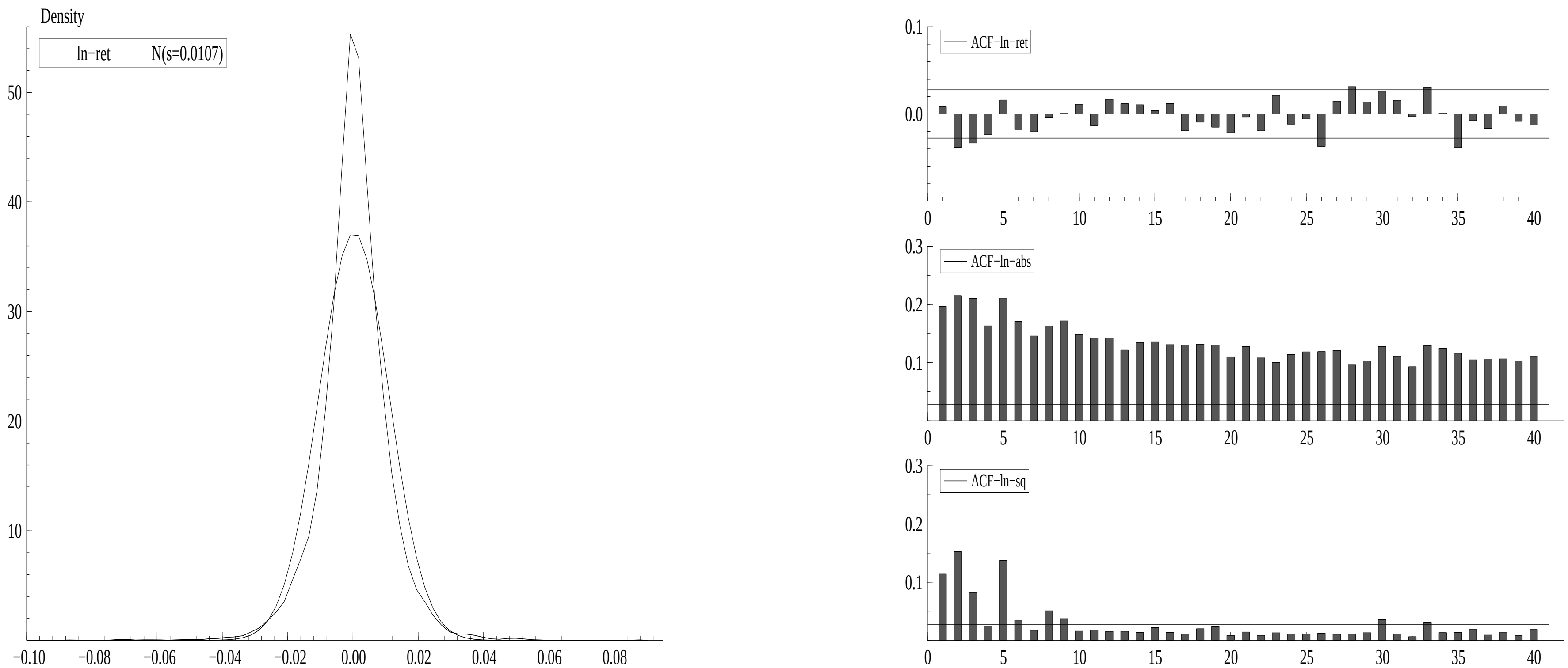


Figure 1: Time series on daily prices and returns, absolute returns, squared returns, return density and their autocorrelation coefficients (ACs) S&P500 from 19/11/1983 to 19/11/2003.

1.2 Econometric Models

- **GARCH—Engle (1982)**
  - **produces fat tails**
  - **captures the short-run dynamics of volatility autocorrelations**
  - **implied decay of the volatility autocorrelation is** *exponential* **rather than** *hyperbolic*
  - **no economic explanation**
- **Power-Law Decay Index and ARFIMA—Baillie (1996)**
- **Volatility clustering and (FI)GARCH —Engle (1982) and Bollersley (1986)**

1.3 Heterogeneity and Bounded Rationality

- **Asset pricing and investor psychology:** *Hirshleifer(2001)*
- **Theoretically oriented—Financial market as nonlinear adaptive evolutionary system:**
  - **Adaptive Rational Beliefs of Heterogeneous Agents and Learning:** *Day and Huang (1990), Chiarella(1992), Brock and Hommes (1997, 98), Hommes (2001), Chiarella and He (2001, 2002)*
  - **Herd Behaviour and the Master Equation:** *Lux (1995)*
  - *Chiarella and He (2002)*—**Risk and Learning Effects.**
- **Computationally oriented:**
  - **Santa Fe stock market: Farmer and Lo (1999), LeBaron (2000)**
  - **Learning: Bullard and Duffy (1999)**
  - **Genetic Algorithms: Chen and Yeh (1997, 2002).**

1.4 Mechanism in Generating Long Memory

- **Herding Model—Alfarano and Lux (2003)**
  - **To incorporate herding and contagion phenomena**
  - **Fundamentalists and noise traders**
  - **Price changes are generated by either exogenous inflow of new information about fundamentals or endogenous changes in demand and supply via the herding mechanism**
  - **The model is able to produce realistic long memory feature**
  - **The underlying mechanism:**
    - **recurrent switches and bi-modal distribution of noise traders in the optimistic and pessimistic groups of individuals;**
    - **the underlying deterministic model displays back and forth movement through a Hopf bifurcation scenario**
  - **However the power-law disappears as popu. size increases**

- **Switching Model—Gaunersdorfer and Hommes (2000)**
  - **Extended Brock and Hommes's Adaptive Belief System**
  - **Nonlinear deterministic model**
  - **Switching between predictors and co-existing attractors**
  - **Adding noise to the deterministic system may trigger switches between low- and high-volatility phases**
  - **Allowing infinite many agents, comparing to herding model**
  - **Comparison with empirical records is mainly based upon visual inspection, or upon a few realizations of the model**
  - **Lacking statistic justification**

1.5 Intuitive Motivations

- **Goals of this literature:**
  - **to model bounded rationality and explain market behaviour and price volatility**
  - **to replicate the econometric properties and stylized facts of financial time series, in particular, of high frequency data**
  - **to understand the generating mechanisms**
  - **to estimate the model**
- *Question*: **How far are we from the goals, in particular for high-frequency modelling?**
- *Answer*:
  - **Promising mathematically**
  - **Unsatisfying statistically**

- *Current issues*:
  - **Unrealistic trading period**
  - **Interplay of noisy and deterministic dynamics**
- **What do we expect?**
  - **Market price should follow the fundamental price when the market is dominated by the fundamental traders.**
  - **Price deviation from the fundamental may be expected when the market is dominated by non-fundamental traders, such as trend followers.**
  - **Excess volatility is expected when the interaction among different trading strategies is high.**
  - **Non-fundamental trading strategies may generate better return over certain short time horizons, but may not be the case in a long-run.**

- **Market Fraction (MF) Model**
  - **A simple stochastic model with fundamentalists and trend followers**
  - **Realistic trading period**
  - **The model achieves what we expected**
  - **Convergence of market price to fundamental value, long/short-run profitability, survivability of chartists and various under/over-reaction autocorrelation patterns** *can be explained* **by the stability and bifurcations of the underlying deterministic system.**
  - **Provides an economic mechanism on the long memory—Heterogeneity, Trend Chasing and Lagged Learning**
- *Approaches*:
  - **Stability and bifurcation analysis for the deterministic model**
  - **Monte Carlo simulations and statistical analysis for the stochastic model**

2 Market Fraction Model via A Market Maker

2.1 Portfolio optimisation problem

- **Notations:**

P_t : Price (ex dividend) per share of the risky asset at time t;

D_t : Dividend at time t;

K : Trading frequency; $K = 250$ (daily);

r : Annual risk free rate ; $R = 1 + r/K$

say for $r = 5\% p.a.$, $R = 1.0002$

$W_{i,t}$: Wealth of agent i at time t;

$W_{i,0}$: Initial wealth of agent i.

- **Portfolio optimization problem**
  - **Assets: one risky asset and one risk-free asset;**
  - **Wealth**

$$\boxed{W_{t+1} = RW_t + [P_{t+1} + D_{t+1} - RP_t]z_t.}$$

  - **Optimal demand $z_{i,t}$:**

$$\boxed{\max_{\pi_{i,t}} E_{i,t}[U(W_{i,t+1})].}$$

where $U_i(W) = -e^{-a_i W}$, $R_{t+1} = P_{t+1} + D_{t+1} - R\,P_t$,

$$\boxed{z_{i,t} = \frac{E_{i,t}(R_{t+1})}{a_i V_{i,t}(R_{t+1})}.}$$

2.2 Heterogeneous Beliefs

- **Fundamentalists:**
  - **Expectation**

$$E_{1,t}(P_{t+1}) = (1-\alpha)P_t + \alpha P^*_{t+1},$$
$$V_{1,t}(P_{t+1}) = \sigma_1^2,$$

  - $\alpha \in [0,1]$ **measures the speed of price adjustment toward the fundamental price, or confidence level of the fundamentalists on the fundamental price.**
  - **The fundamental price** P_t^***:**

$$P^*_{t+1} = P^*_t[1 + \sigma_\epsilon \tilde{\epsilon}_t], \qquad \tilde{\epsilon}_t \sim \mathcal{N}(0,1).$$

- **Trend followers:**
  - **Expectation:**

$$E_{2,t}(P_{t+1}) = P_t + \gamma(P_t - u_t),$$
$$V_{2,t}(P_{t+1}) = \sigma_1^2 + b_2 v_t,$$

where γ measures the extrapolation rate.

  - **Sample mean and variance**

$$u_t = \delta u_{t-1} + (1-\delta)P_t,$$
$$v_t = \delta v_{t-1} + \delta(1-\delta)(P_t - u_{t-1})^2.$$

are assumed to follow geometric decay probability process:

$$\text{Price history :}\{P_t, P_{t-1}, P_{t-2}, \cdots\}$$
$$\text{Probability :}(1-\delta)\{1, \delta, \delta^2, \cdots\},$$

– δ: memory parameter measures the geometric decay rate.

  * For $\delta = 0$, the sample mean $u_t = P_t$ is the latest observed price;
  * For $\delta = 0.1$ gives a half life of 0.43 day;
  * For $\delta = 0.5$ gives a half life of 1 day;
  * For $\delta = 0.95$ gives a half life of 2.5 weeks;
  * For $\delta = 0.999$ gives a half life of about 2.7 years.

2.3 Market Equilibrium via a Market Maker

- **Market fractions: fundamentalists n_1 and trend followers n_2**
- **Denote $m = n_1 - n_2$.**
- **The excess demand**

$$z_{e,t} \equiv \frac{1+m}{2} \frac{E_{1,t}[R_{t+1}]}{a_1 V_{1,t}[R_{t+1}]} + \frac{1-m}{2} \frac{E_{2,t}[R_{t+1}]}{a_2 V_{2,t}[R_{t+1}]}.$$

- **The market maker takes a long (when $z_{e,t} < 0$) or short (when $z_{e,t} > 0$) position so as to clear the market and adjusts the price :**

$$P_{t+1} = P_t + \mu z_{e,t} + \tilde{\delta}_t,$$

where μ measures speed of price adjustment, $\tilde{\delta}_t \sim N(0, \sigma_\epsilon^2)$ captures a randomly unexpected demand noise process.

- **The complete stochastic model:**

$$
\begin{cases}
P_{t+1} = P_t + \dfrac{\mu}{2}\Big[\dfrac{1+m}{a_1(1+q)\sigma_1^2}[\alpha(P^*_{t+1} - P_t) - (R-1)(P_t - \bar{P})] \\
\qquad\qquad + (1-m)\dfrac{\gamma(P_t - u_t) - (R-1)(P_t - \bar{P})}{a_2\sigma_1^2(1+q+b\,v_t)}\Big] + \tilde{\delta}_t, \\
u_t = \delta u_{t-1} + (1-\delta)P_t, \\
v_t = \delta v_{t-1} + \delta(1-\delta)(P_t - u_{t-1})^2, \\
P^*_{t+1} = P^*_t[1 + \sigma_\epsilon \tilde{\epsilon}_t].
\end{cases}
$$

2.4 Wealth Dynamics

- **Auxiliary functions:**

$$V_{1,t} = 1/W_{1,t}, \qquad V_{2,t} = 1/W_{2,t}.$$

Then

$$\begin{cases} V_{1,t+1} = \dfrac{V_{1,t}}{R + R_{t+1} z_{1,t} V_{1,t}}, \\ V_{2,t+1} = \dfrac{V_{2,t}}{R + R_{t+1} z_{1,t} V_{1,t}}. \end{cases}$$

- *Absolute wealth proportion*:

$$w_{1,t} = \frac{W_{1,t}}{W_{1,t} + W_{2,t}}, \qquad w_{2,t} = \frac{W_{2,t}}{W_{2,t} + W_{2,t}}.$$

- *Market wealth proportion*:

$$\begin{cases} \bar{w}_{1,t} = \dfrac{(1+m)W_{1,t}}{(1+m)W_{1,t} + (1-m)W_{2,t}}, \\ \bar{w}_{2,t} = \dfrac{(1-m)W_{2,t}}{(1+m)W_{1,t} + (1-m)W_{2,t}}. \end{cases}$$

3 Deterministic System—Stability and Bifurcation Analysis

- $(P_t, u_t, v_t) = (\bar{P}, \bar{P}, 0)$ is the unique steady state of the system.
- for $\delta = 0$, the fundamental steady state is stable for $0 < \mu < \mu^*$. , where
$$\mu^* = \frac{2Q}{(R-1)(1-m) + a(R-\alpha)(1+m)}.$$
 In addition, a flip bifurcation occurs along the boundary $\mu = \mu^*$ with $\alpha \in [0, 1]$;
- for $\delta \in (0, 1)$, the fundamental steady state is stable for

$$0 < \mu < \begin{array}{ll} \mu_1 & 0 \leq \gamma \leq \gamma_0 \\ \mu_2, & \gamma_0 \leq \gamma, \end{array}$$

where

$$\mu_1 = \frac{1+\delta}{\delta}\frac{Q}{1-m}\frac{1}{\gamma_2-\gamma},$$

$$\mu_2 = \frac{1-\delta}{\delta}\frac{Q}{1-m}\frac{1}{\gamma-\gamma_1},$$

$$\gamma_1 = (R-1)+a(R-\alpha)\frac{1+m}{1-m},$$

$$\gamma_0 = \frac{(1+\delta)^2}{4\delta}\gamma_1, \qquad \gamma_2 = \frac{1+\delta}{2\delta}\gamma_1.$$

In addition,

– **a flip bifurcation occurs along $\mu = \mu_1$ for $0 < \gamma \leq \gamma_0$ and**

– **a Hopf bifurcation occurs along $\mu = \mu_2$ for $\gamma \geq \gamma_0$.**

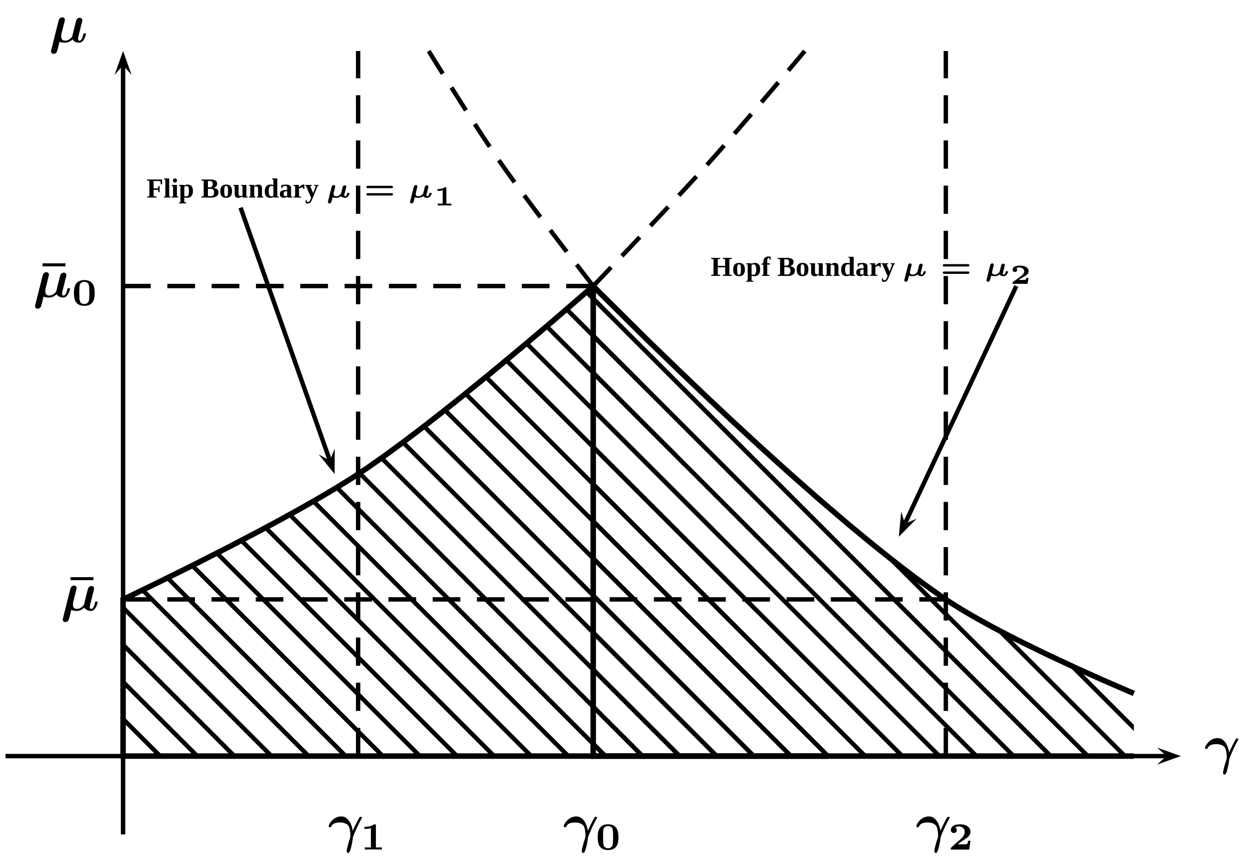


Figure 2: Stability region and bifurcation boundaries for $m \in (-1, 1)$ and $\delta \in (0, 1)$.

- **Implications:**
  - *The market fraction* m **has a great impact on the stability:**
    - **The stability region is enlarged as the fraction of the fundamentalists increases**
    - **The flip bifurcation boundary becomes dominant as the fraction of the fundamentalists increases**
    - **The Hopf bifurcation boundary becomes dominant as the fraction of the trend followers increases**
  - *The speed of price adjustment of the fundamentalists towards the fundamental value* **has a similar impact to the market fraction**
  - *The memory decay rate* **of the trend followers has a similar impact on the price behavior to the market fraction.**

4 Statistical Analysis of the Stochastic Model

We choose annual volatility of the fundamental price to be 20% p.a., $K = 250$, and the volatility of the noisy demand $\sigma_\delta = 1$, which is about 1% of the average fundamental price level $\bar{P} = \$100$.

4.1 Random Fixed Point and Limiting Behaviour

- **A random fixed point: a stationary solution of a stochastic difference system**
- **The asymptotic stability: sample paths converge to the random fixed point point wise for all initial conditions of the system**

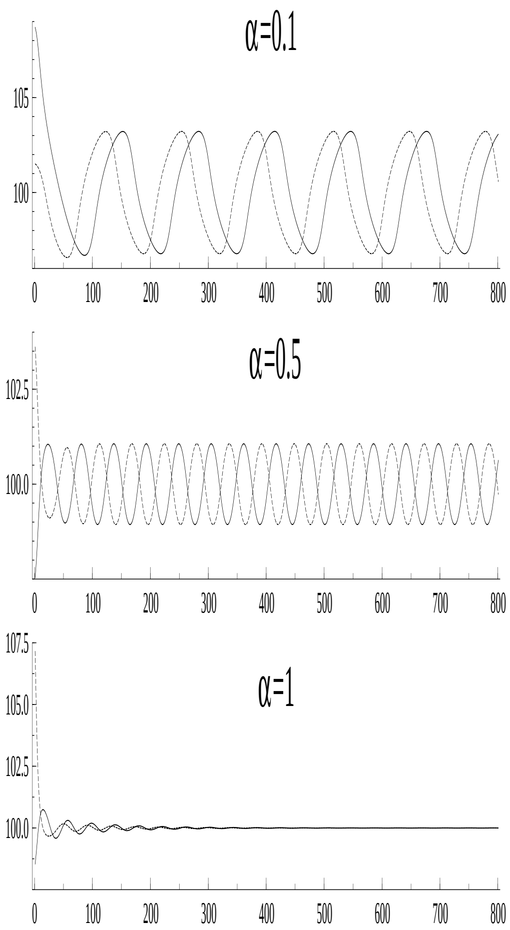
- **We are interested in the existence and stability of a random fixed point of SDS when the deterministic attractor, e.g. fixed point, of DDS is asymptotical stable.**
- **Parameters:**

$$\gamma = 2.1, \delta = 0.85, \mu = 0.43, m = 0, w_{1,0} = 0.5,$$

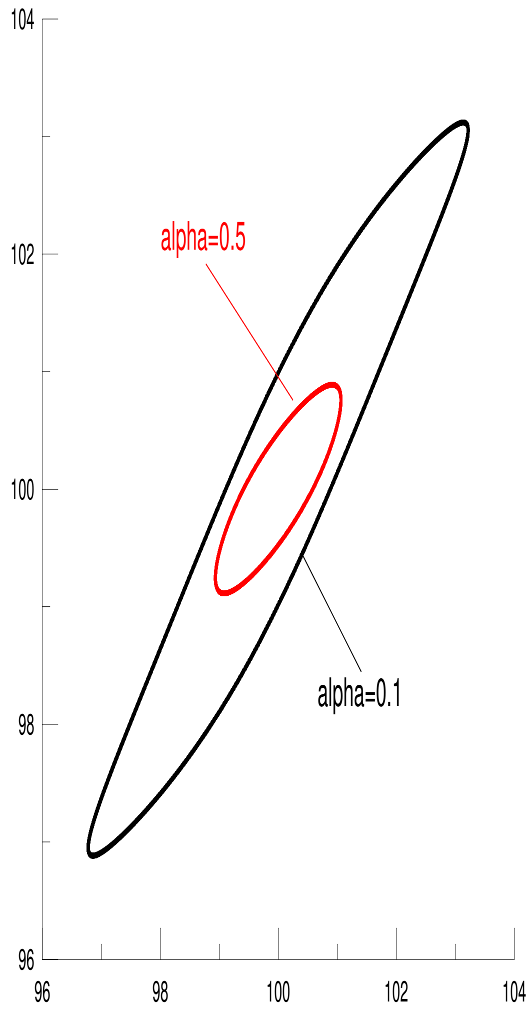
$$\sigma_\epsilon = 20\% p.a., \sigma_\delta = 1, \qquad \alpha = 1, 0.5, 0.1, 0.$$

- **Surprising results:**
  - *Stabilizing effect of the noisy process*—**for $\alpha = 0$, the price explodes for DDS and converges to a stationary distribution for SDS**
  - **The stable attractors of the DDS correspond to the stable random fixed points.**

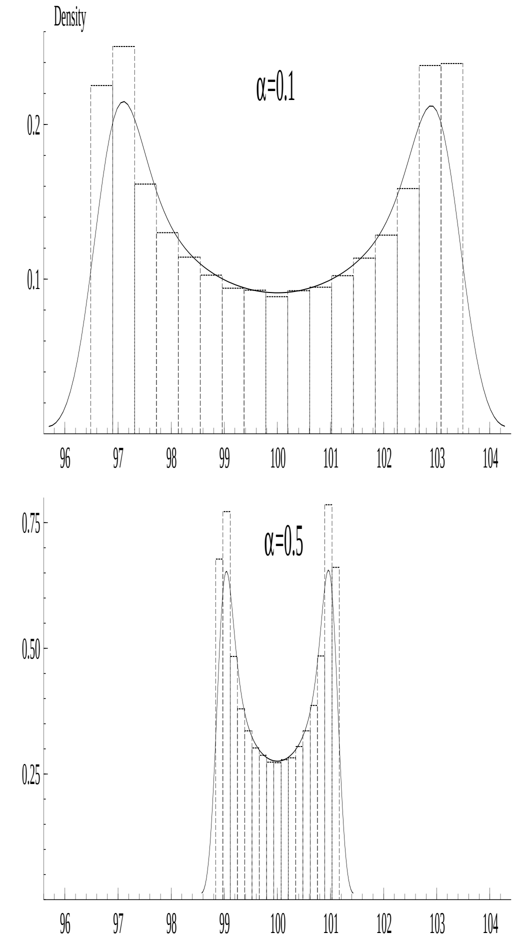
- **Limiting distributions:**
  - **The market price distributions look very similar to the one for the fundamental price for $\alpha = 1, 0.9, 0.5, 0.1$, but different for $\alpha = 0$;**
  - **The return distributions for $\alpha = 1, 0.9, 0.5, 0$ are very different from that for the fundamental return**



α=0.1
α=0.5
α=1



alpha=0.5
alpha=0.1



Density
α=0.1
α=0.5

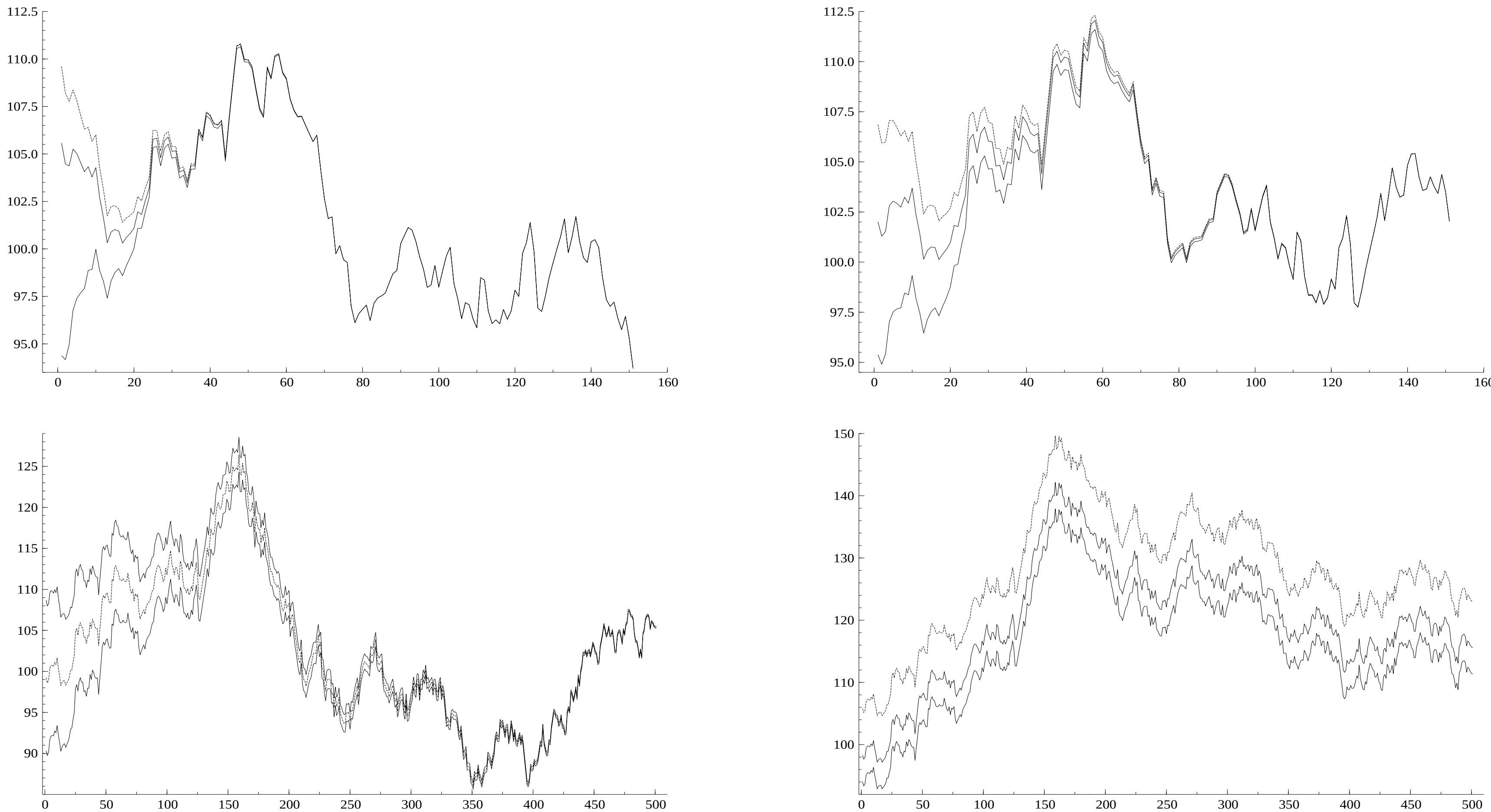
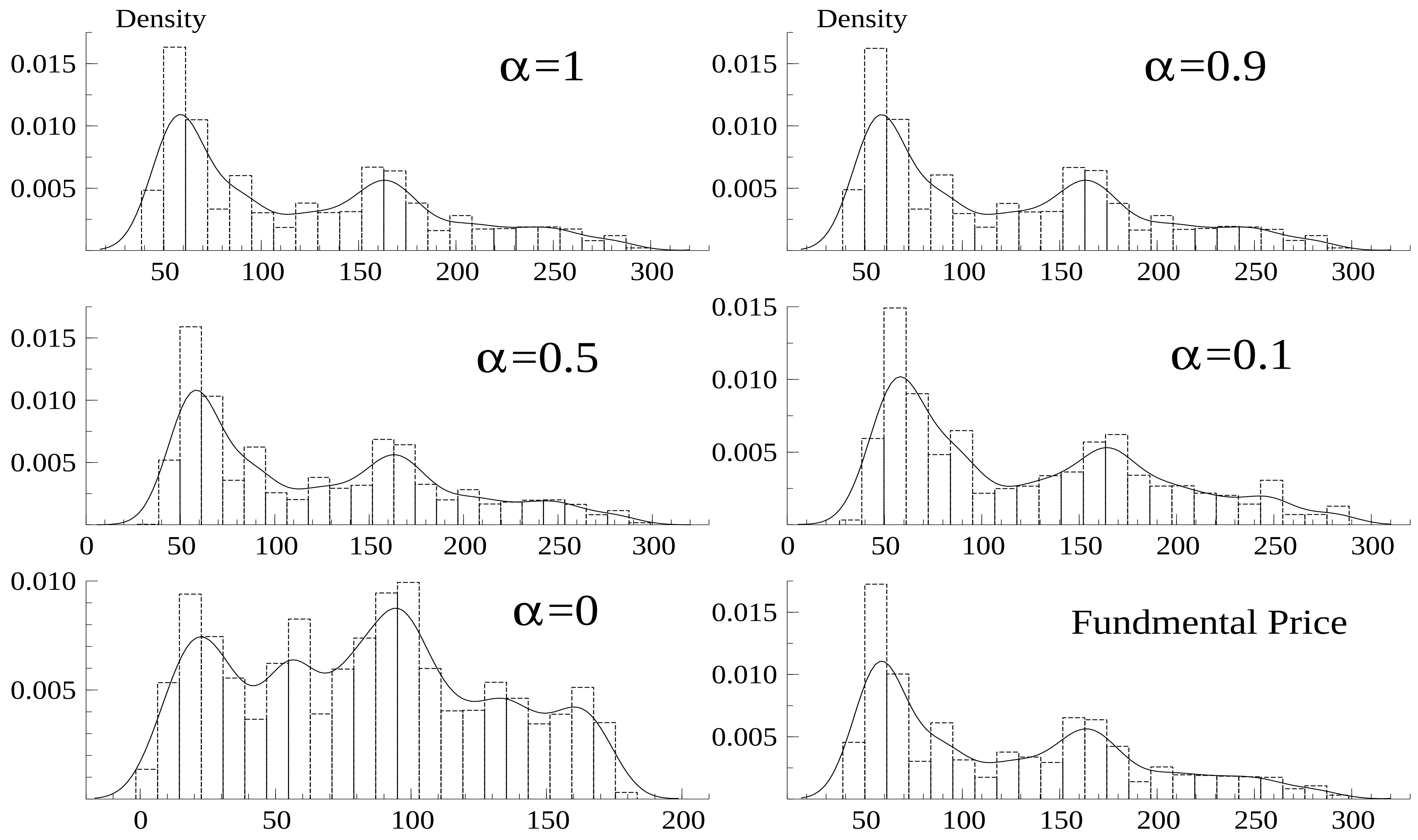
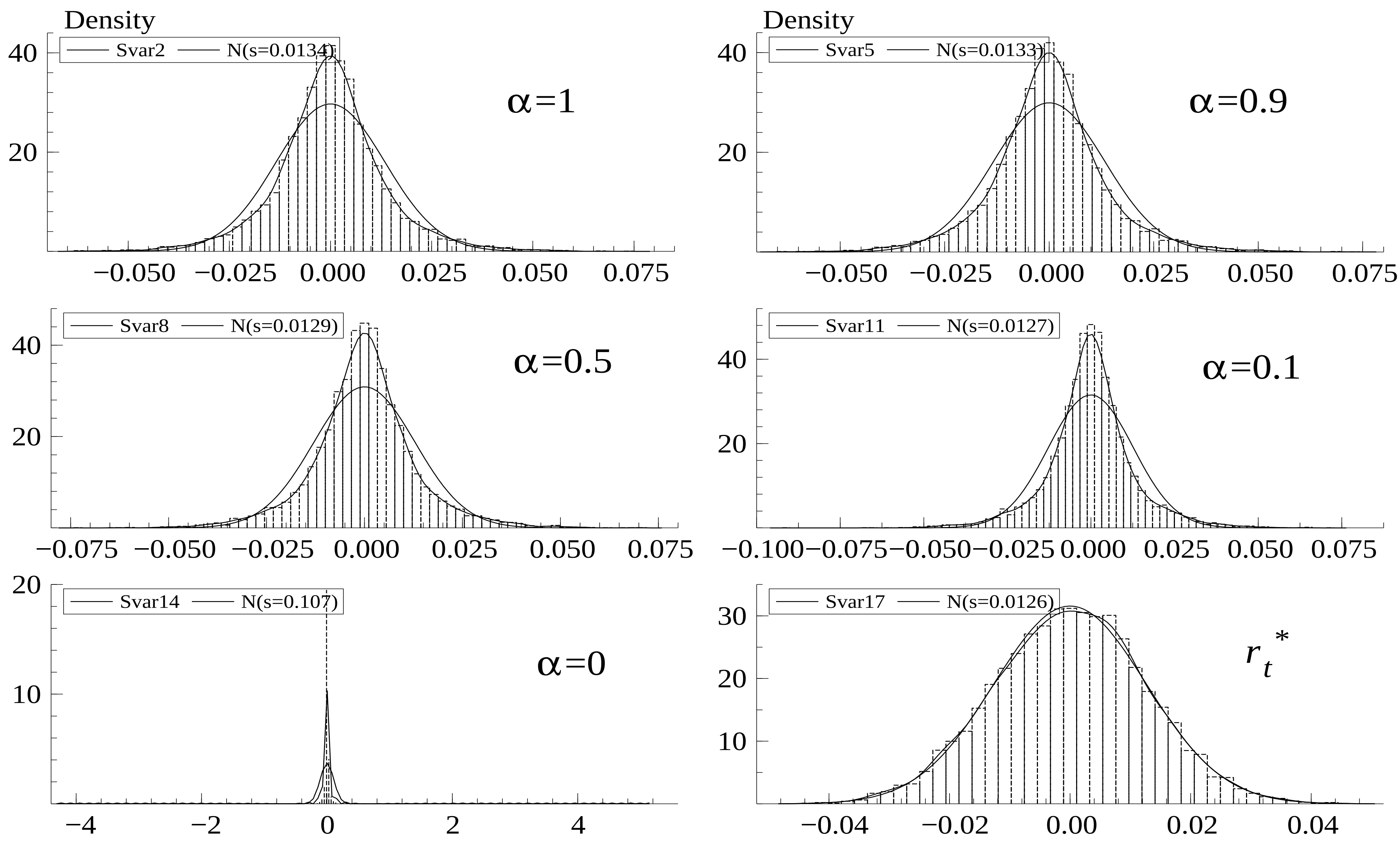


Figure 3: Price convergence with α=1 (a); 0.5 (b); 0.1 (c); and 0 (d) for different initial conditions.


Density
α=1
0.015
0.010
0.005
50 100 150 200 250 300
Density
α=0.9
α=0.5
0
α=0.1
α=0
200
Fundmental Price


Density
Svar2
N(s=0.0134)
α=1
40
20
−0.050
−0.025
0.000
0.025
0.050
0.075
Density
Svar5
N(s=0.0133)
α=0.9
40
20
−0.050
−0.025
0.000
0.025
0.050
0.075
Svar8
N(s=0.0129)
α=0.5
40
20
−0.075
−0.050
−0.025
0.000
0.025
0.050
0.075
Svar11
N(s=0.0127)
α=0.1
40
20
−0.100
−0.075
−0.050
−0.025
0.000
0.025
0.050
0.075
Svar14
N(s=0.107)
α=0
20
10
−4
−2
0
2
4
Svar17
N(s=0.0126)
r_t^*
30
20
10
−0.04
−0.02
0.00
0.02
0.04

4.2 Convergence of Market Price to the Fundamental Value

- **The null hypothesis to detect the differences between market prices and fundamental prices is specified as, respectively,**
  - Case 1: $H_0 : P_t = P_t^*, t = 1000, 2000, \ldots, 5000;$
  - Case 2, $H_0 : P_t = P_t^*, t = 3000, 3500, 4000, \ldots, 5000;$
  - Case 3, $H_0 : P_t = P_t^*, t = 4000, 4100, 4200, \ldots, 5000;$
  - Case 4, $H_0 : P_t = P_t^*, t = 4000, 4050, 4100, \ldots, 5000;$
  - Case 5, $H_0 : P_t = P_t^*, t = 4901, 4902, 4903 \ldots, 5000$, which refers to the last 100 periods;
  - Case 6, $H_0 : P_t = P_t^*, t = 4951, 4952, \ldots, 5000$, which refers to the last 50 periods.

- **The resulting Wald statistics:**

	$\alpha = 0$	$\alpha = 0.1$	$\alpha = 0.5$	$\alpha = 1$	Critical value
Case 1	100.585	13.289	5.225	3.698	11.071
Case2	99.817	13.964	6.782	4.358	11.071
Case 3	121.761	24.971	16.041	10.840	19.675
Case 4	148.690	38.038	23.836	19.190	32.671
Case 5	293.963	105.226	99.618	103.299	124.342
Case 6	177.573	50.970	45.043	43.052	67.505

- **Results:**
  - **For $\alpha = 0.5$ and 1, all of the null hypothesis cannot be rejected at the 5% significant level.**
  - **When α increases, the resulting Wald statistics decreases (except Case 5 with $\alpha = 1$), i.e. when the fundamentalists put more weight on the fundamental price, the differences between the market prices and fundamental prices become smaller.**

4.3 Wealth Accumulation, Profitability and Survivability

- *The impact of* α**:**

$$\gamma = 2.1,\ \delta = 0.85,\ \mu = 0.43,\ m = 0,\ w_{1,0} = 0.5,$$

$$\alpha = 1,\ 0.5,\ 0.1,\ 0$$

  - **trend followers survive in the long-run for $\alpha = 1, 0.5$ and 0.1, although they accumulate less wealth shares over the time period;**
  - **the trend followers do better than the fundamentalists when $\alpha = 0$;**
  - **the profitability of the fundamentalists improves as α increases.**

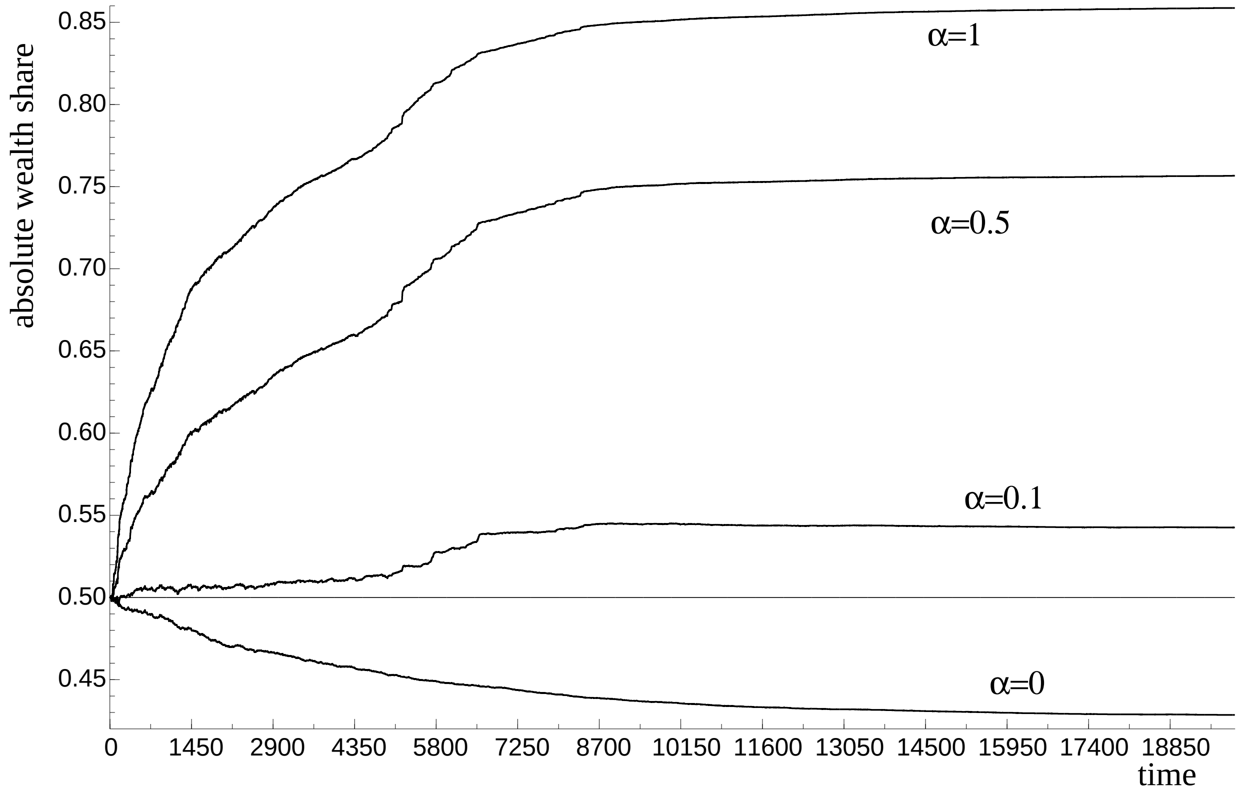
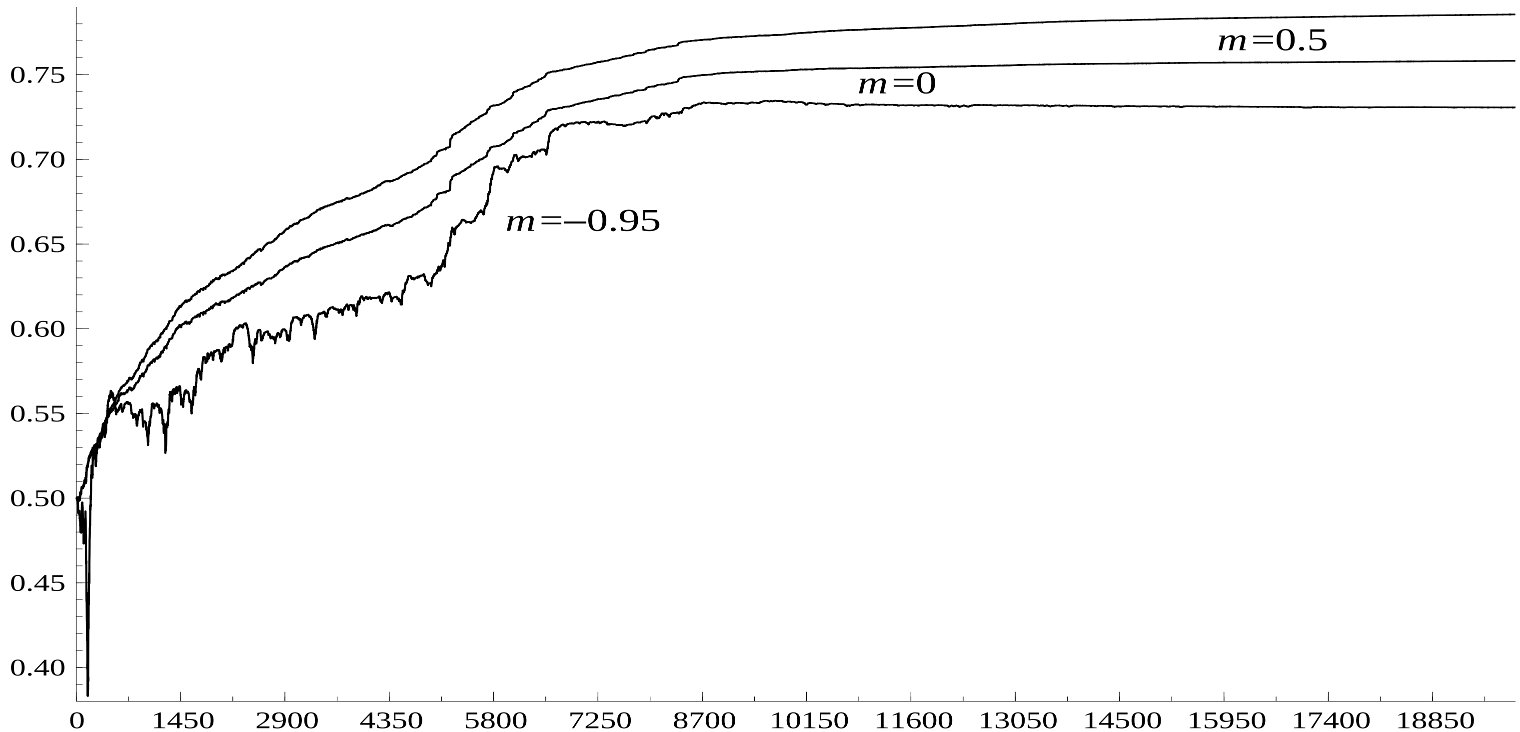
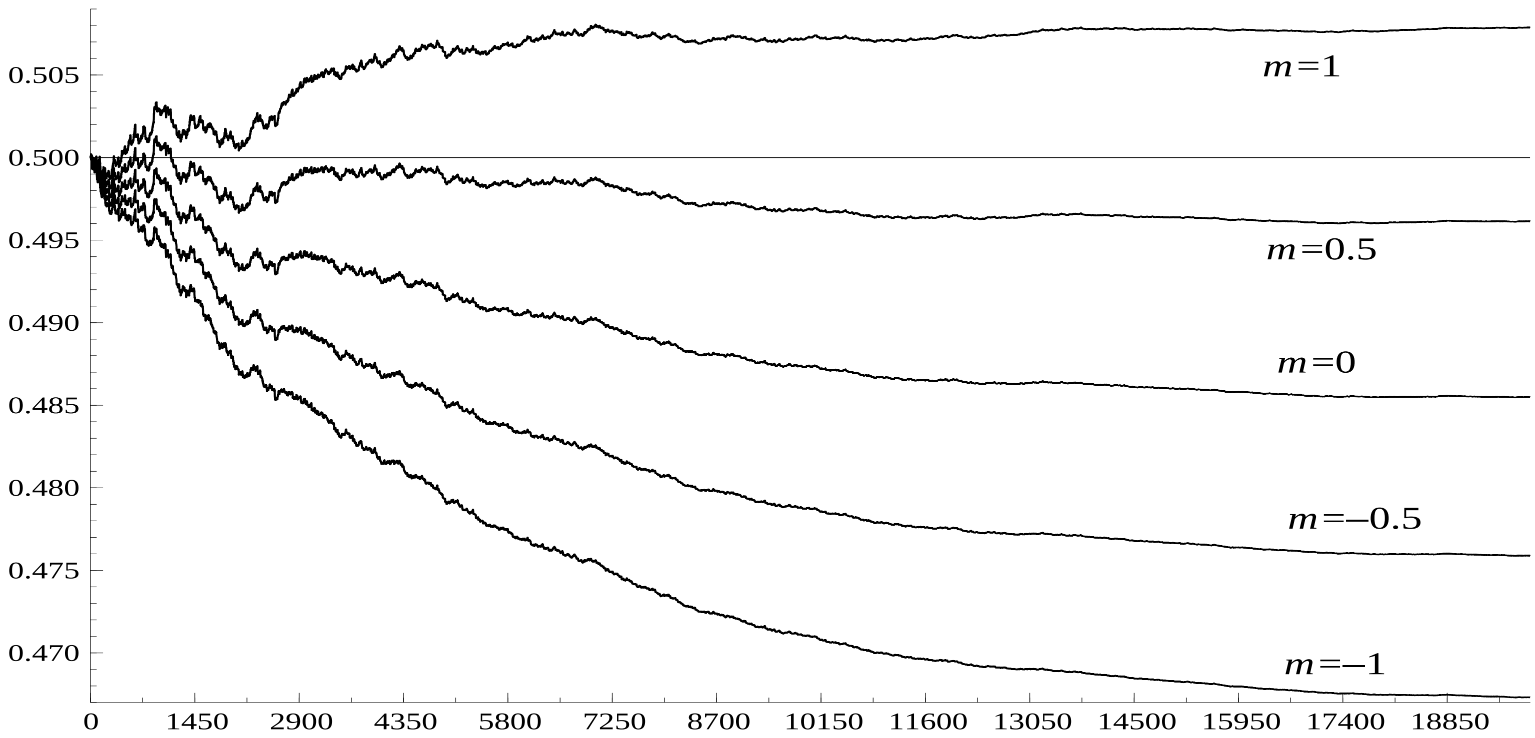


Figure 4: Time series of the absolute wealth accumulation of the fundamentalists $w_{1,t}$ with $\alpha = 1,\ 0.5,\ 0.1$ and 0.

- *The impact of m*: **The absolute wealth share accumulations of the fundamentalists for three different values of $m = -0.95, 0$ and 0.5 with $\alpha = 0.5, \gamma = 2, \mu = 0.5, \delta = 0.85, w_{1,0} = 0.5$**



- **The absolute wealth share accumulations of the fundamentalists for three different values of $m = -1, -0.5, 0, 0.5, 1$ and $\alpha = 0, \gamma = 1, \mu = 0.4, \delta = 0.85, w_{1,0} = 0.5$.**



- **Implications:**
  - **The trend followers survive in a long-run, can do even better in a short-run due to the learning.**
  - **The profitability of the fundamentalists improves as either they become more confident on their estimated fundamental value or they dominate the market.**
  - **The trend followers are doing better by accumulating a higher wealth share when the fundamentalists become naive traders. In addition, their profitability improves as their market population share increases.**

4.4 Bifurcations and Autocorrelation Patterns

- **Understanding the autocorrelation (AC) structure of returns plays an important role in the market efficiency and predictability.**
- **We believe that the underlying deterministic dynamics near the bifurcation boundaries play an important role in the AC structure of the stochastic system.**
- **Monte Carlo simulations: Two sets of parameters near flip and Hopf bifurcation boundaries:** $N = 1,000$

(F1) $\alpha = 1, \gamma = 0.8, \mu = 5, \delta = 0.85, w_{1,0} = 0.5$ **and** $m = -0.8, -0.5, -0.3, 0;$

(H1) $\alpha = 1, \gamma = 2.1, \mu = 0.43, \delta = 0.85, w_{1,0} = 0.5$ **and** $m = -0.95, -0.5, 0, 0.5.$

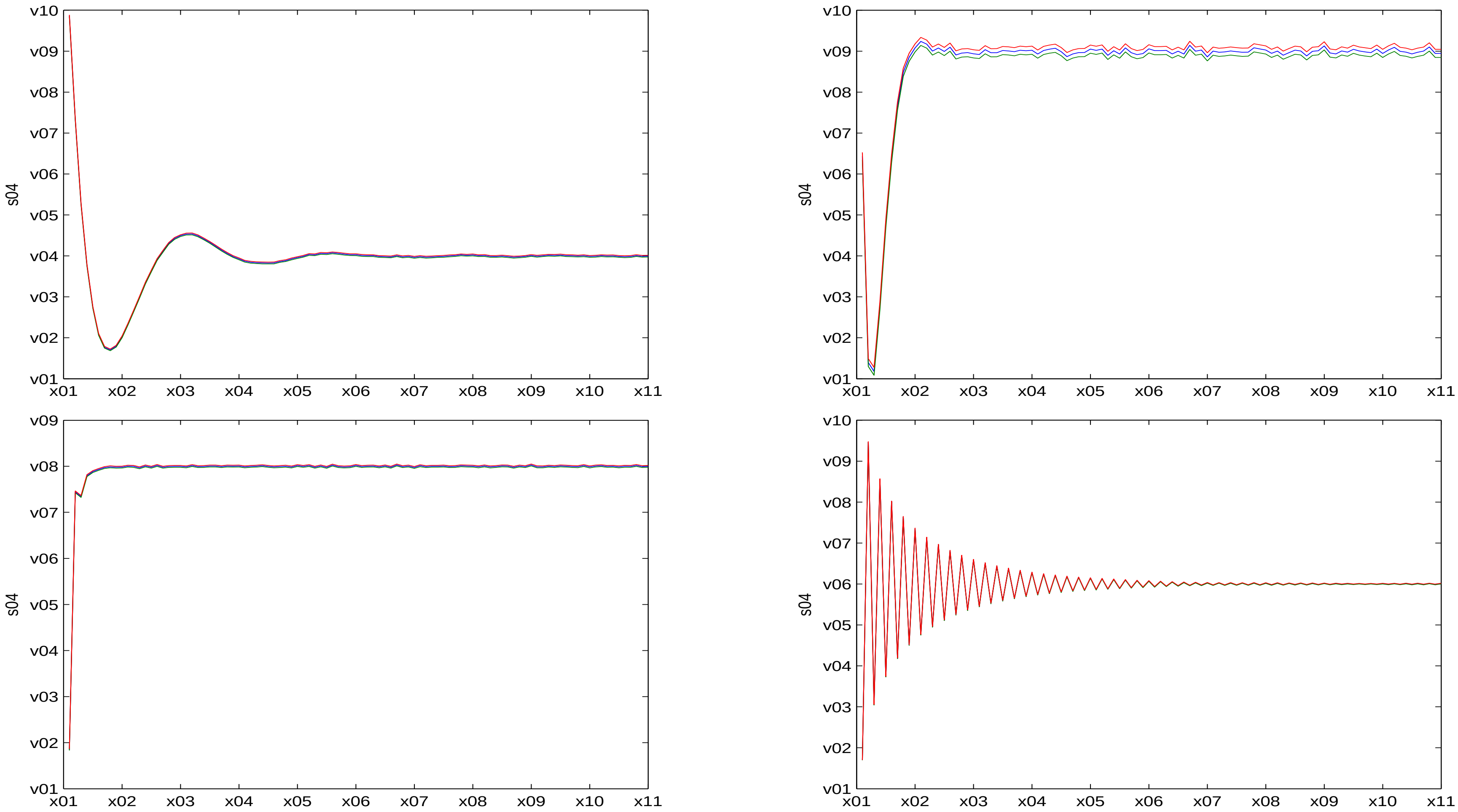


Figure 5: Monte Carlo simulation on the average ACs of return for $m = -0.8, -0.5, -0.3, 0$ for parameter set $(F1)$.

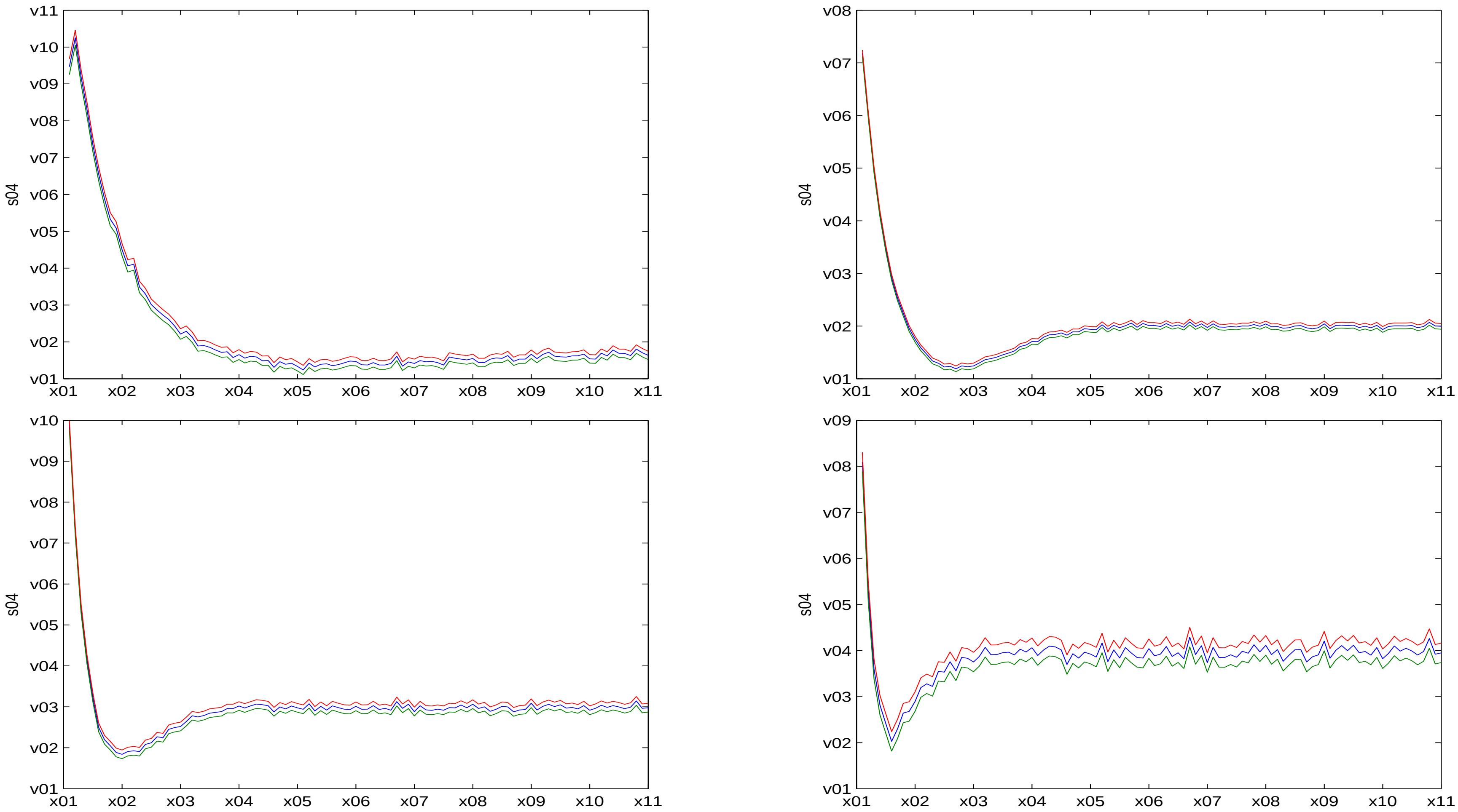


Figure 6: Monte Carlo simulation on the average ACs of return for $m = -0.95, -0.5, 0, 0.5$ for parameter set $(H1)$.

- **Implications:**
  - **Activity of the fundamentalists (either high fraction or high speed of price adjustment) are responsible for over-reaction AC patterns and extrapolation from the trend followers are responsible for the under-reaction AC patterns.**
  - **A strong under-reaction AC patterns of SDS is in general associated with Hopf bifurcation of the DDS.**
  - **A strong over-reaction AC pattern is associated with flip bifurcation,**
  - **Short-run under and long-run over-reaction AC patterns are associated with both types of bifurcation (depending on their dominance).**

5 Mechanism Analysis on Long Memory

- **Aims:**
  - **To characterise the interplay between system size, deterministic forces and stochastic elements**
  - **To find potential mechanism in generating realistic time series properties, in particular, the long memory**
- **Parameters:**

α	γ	a_1	a_2	μ	m	δ	b	σ_ϵ	σ_δ	P_0^*
0.1	0.3	0.8	0.8	2	0	0.85	1	0.01265	1	100

- **Four cases:**

Cases	Case-00	Case-01	Case-10	Case-11
$(\sigma_\delta, \sigma_\epsilon)$	$(0, 0)$	$(0, 0.01265)$	$(1, 0)$	$(1, 0.01265)$

- **Time Series:**
  - **Market Price**

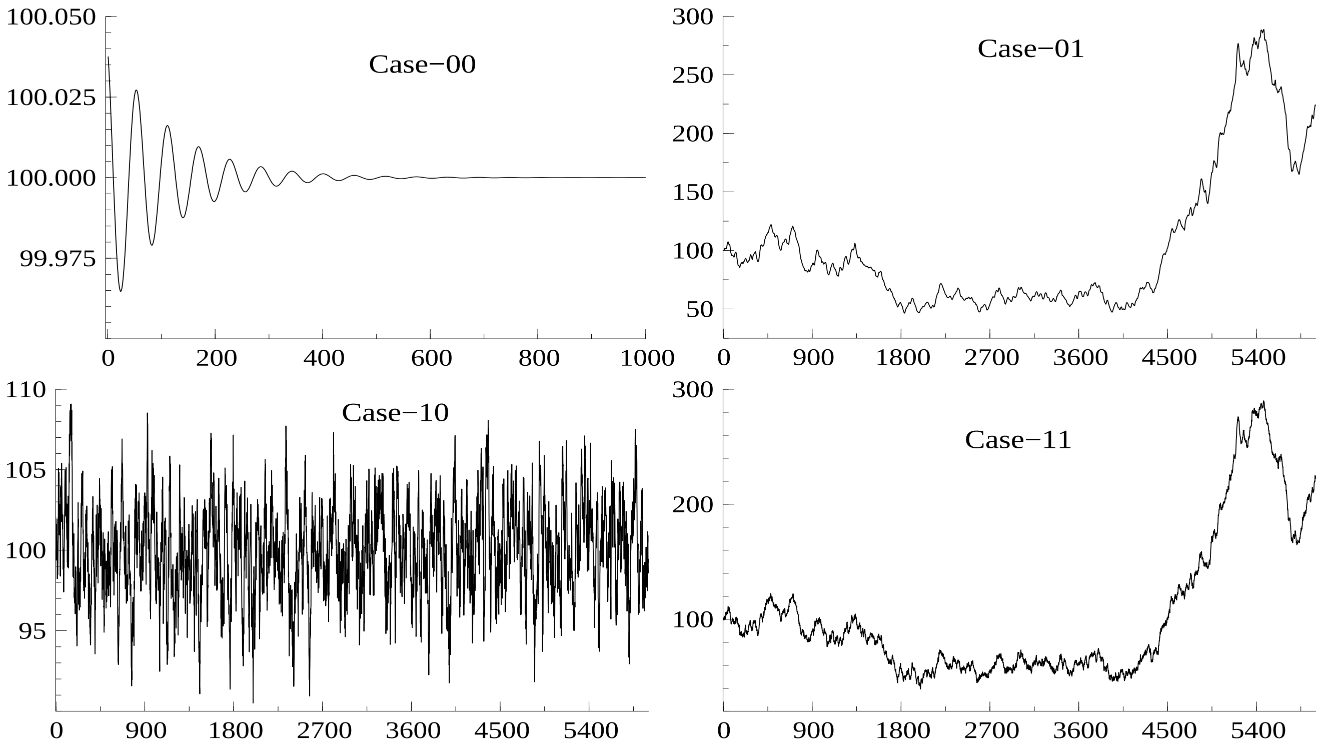


Figure 7: Time series of prices for four cases.

– Return

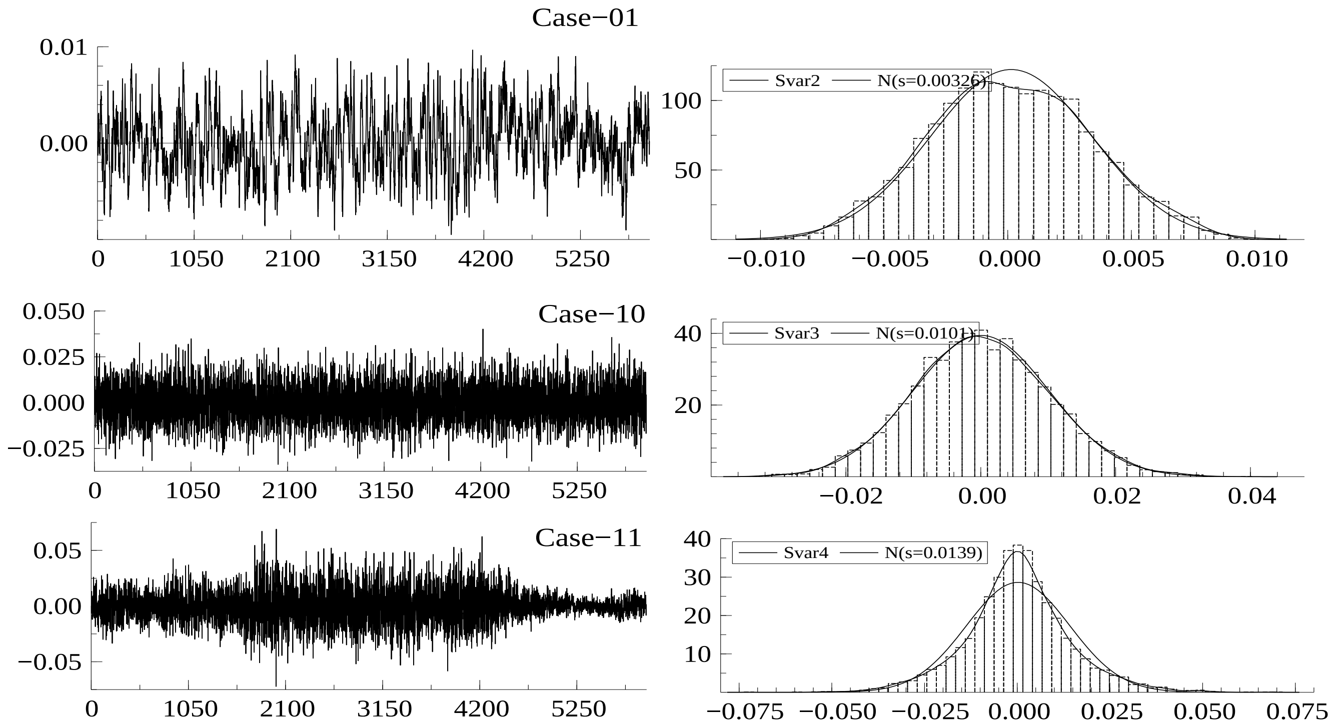


Figure 8: Time series and density distributions of the returns of Cases-01, 10 and 11.

- **Autocorrelations:**

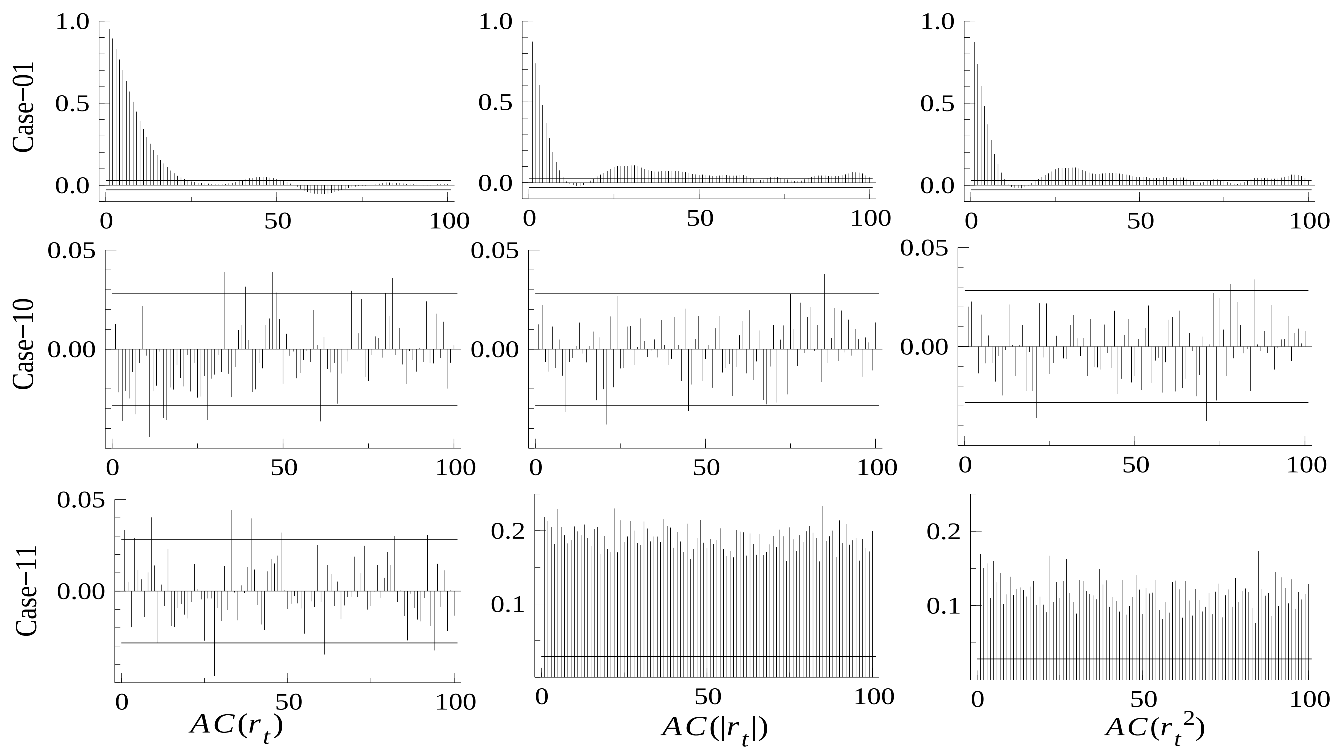


Figure 9: The ACs of returns (left column), the absolute returns (middle column), and the squared returns (right column) of Cases-01, 10 and 11.

- **Fundamental Price:**

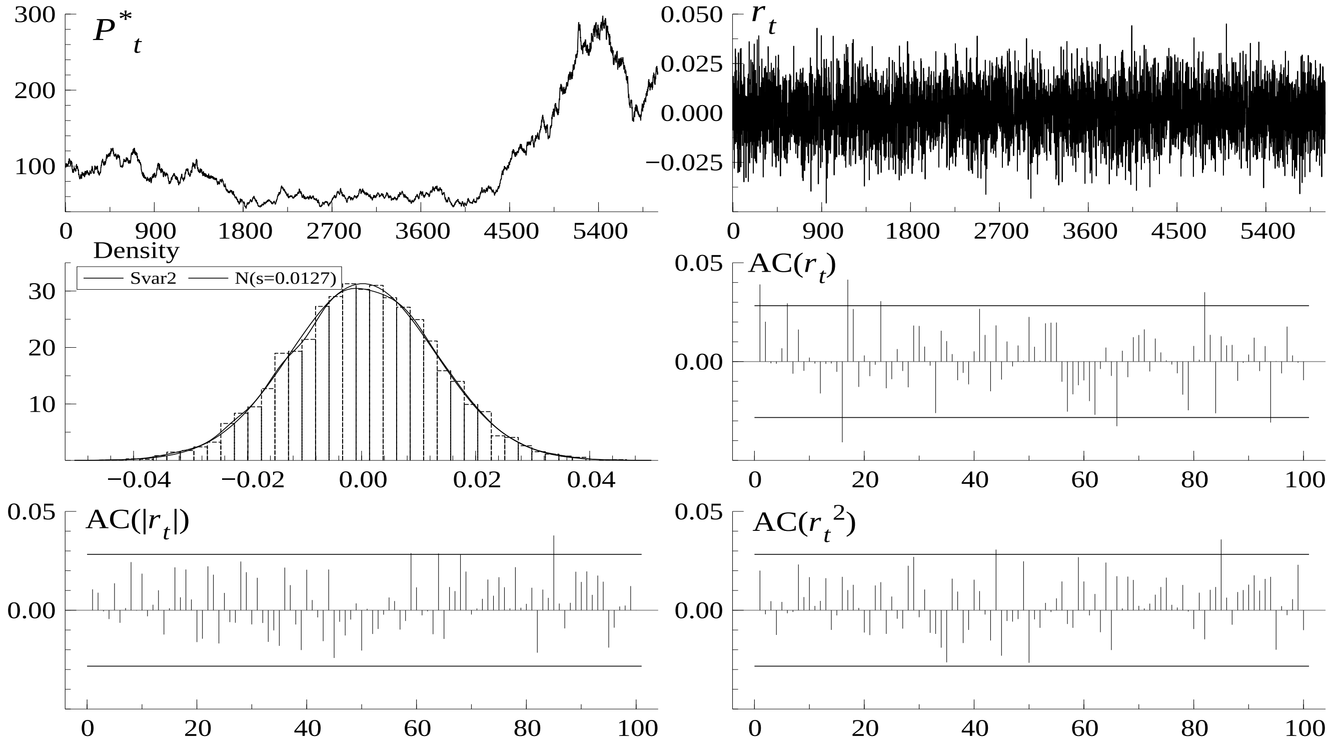


Figure 10: Time series of the fundamental price and return, the return distribution density and the corresponding ACs of returns, the absolute returns, and the squared returns.

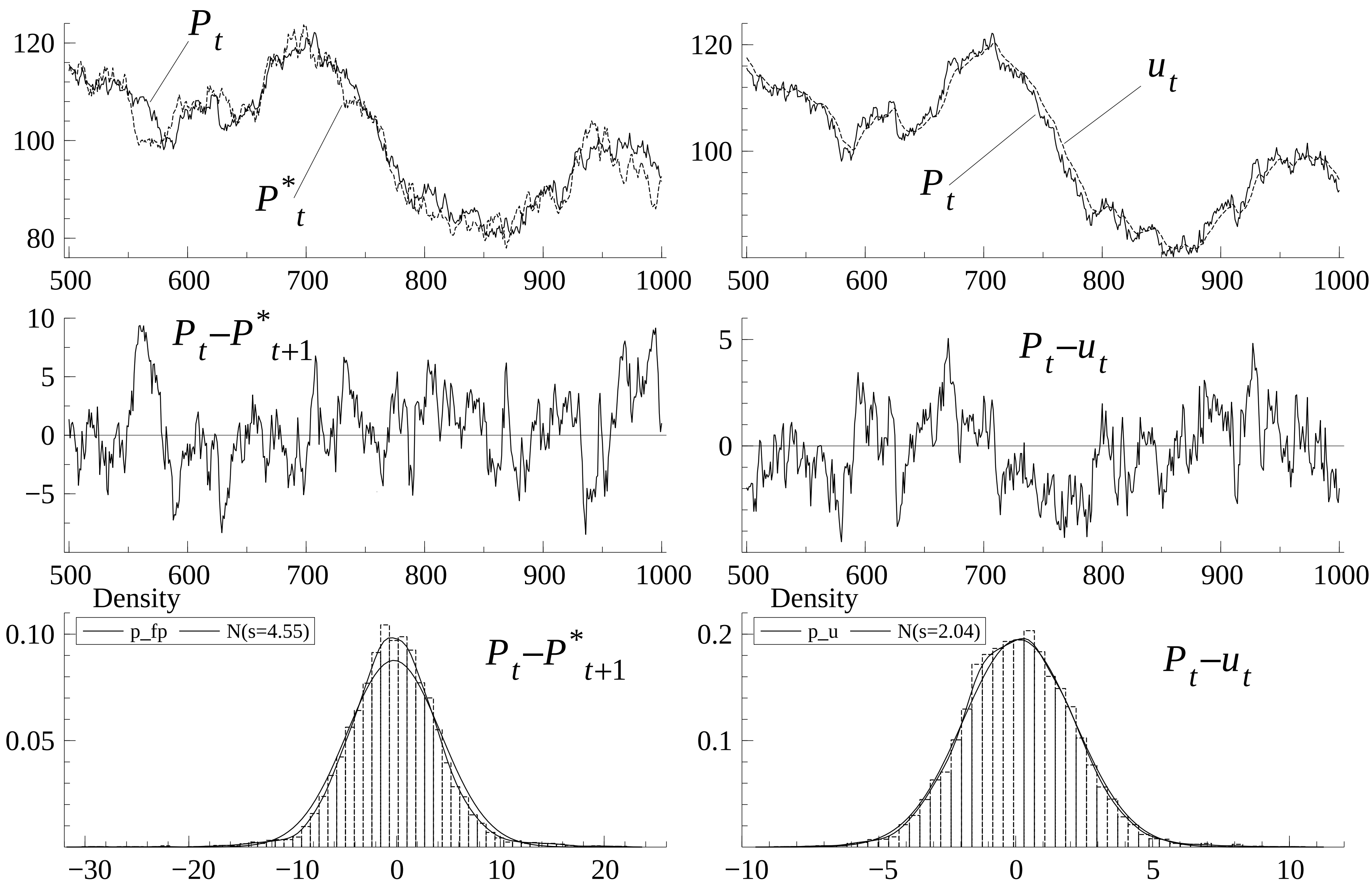
- *Findings*: **The simple MF model appears to do the job very well when both noisy processes present**

- **Explanation—Case-00**
  - **The trend follows a geometric decay process and is updated upon history market price**
  - **The learning process smooths the price and leads to a lagged reaction to the market price**
  - **Mathematically, an extrapolation of the trend followers towards the trend leads to Hopf bifurcation and the choose parameter γ is near the Hopf bifurcation boundary**
  - **Intuitively, the nature of the oscillating convergence to the steady state is due to the extrapolation and learning of the trend followers.**
  - **It is this** *lagged learning and trend chasing* **that plays important role for the dependent volatility**

- **Explanation—Case-01**
  - **the stochastic fundamental price fluctuates lead to recurrent shifts of the fundamental values to different levels; too often to leave the trend followers no enough time to learn the true fundamental value**
  - **the lagged learning from the trend followers leads to highly dependent volatility (measured by the absolute and squared returns) over short-run**
- **Explanation—Case-10:**
  - **The lagged learning does not prevent trend followers from learning the constant fundamental values**
  - **the return is close to normal and there is no significant AC patterns for return, the absolute return and squared return**

- **Explanation—Case-11**
  - **the stochastic nature from the noisy excess demand and the weak extrapolation from the trend followers prevent the market price from forming any significant trend, leading to no significant AC pattern for returns**
  - **However, the volatility fluctuations due to the lagged learning from the trend followers are carried on**
  - **Because of the stochastic nature of the noisy excess demand, the strong AC patterns of the absolute and squared returns shown in Case-01 are washed out, but still highly significant**
  - **It is worth emphasizing that neither one of the two noisy processes alone is responsible for this realistic feature.**

- **The impact of the noise processes on the market price and its relation to the fundamental price.**
  - **The market price moves closely to the fundamental price;**
  - **Temporary deviation of the market price from the fundamental price can be significant from time to time.**
  - **The market prices are more concentrated near the fundamental prices most of the time.**
  - **The moving averaged price is less volatile. Also, its difference from the market price is less concentrated near zero.**


P_t
P^*_t
u_t
P_t
$P_t - P^*_{t+1}$
$P_t - u_t$
Density
p_fp N(s=4.55)
$P_t - P^*_{t+1}$
Density
p_u N(s=2.04)
$P_t - u_t$

- *Volatility clustering and the endogenous learning process*
  - **A simple feedback effect—The trend followers tend to push the market price away from the fundamental value by extrapolating the trend, leading to high volatility. Because of the perceived increase of risk, their demand/supply is then reduced. The partial withdrawal of the trend followers then leads to less volatile dynamics, which makes the trend followers revise the risk downward so that eventually their demand/supply increases again.**

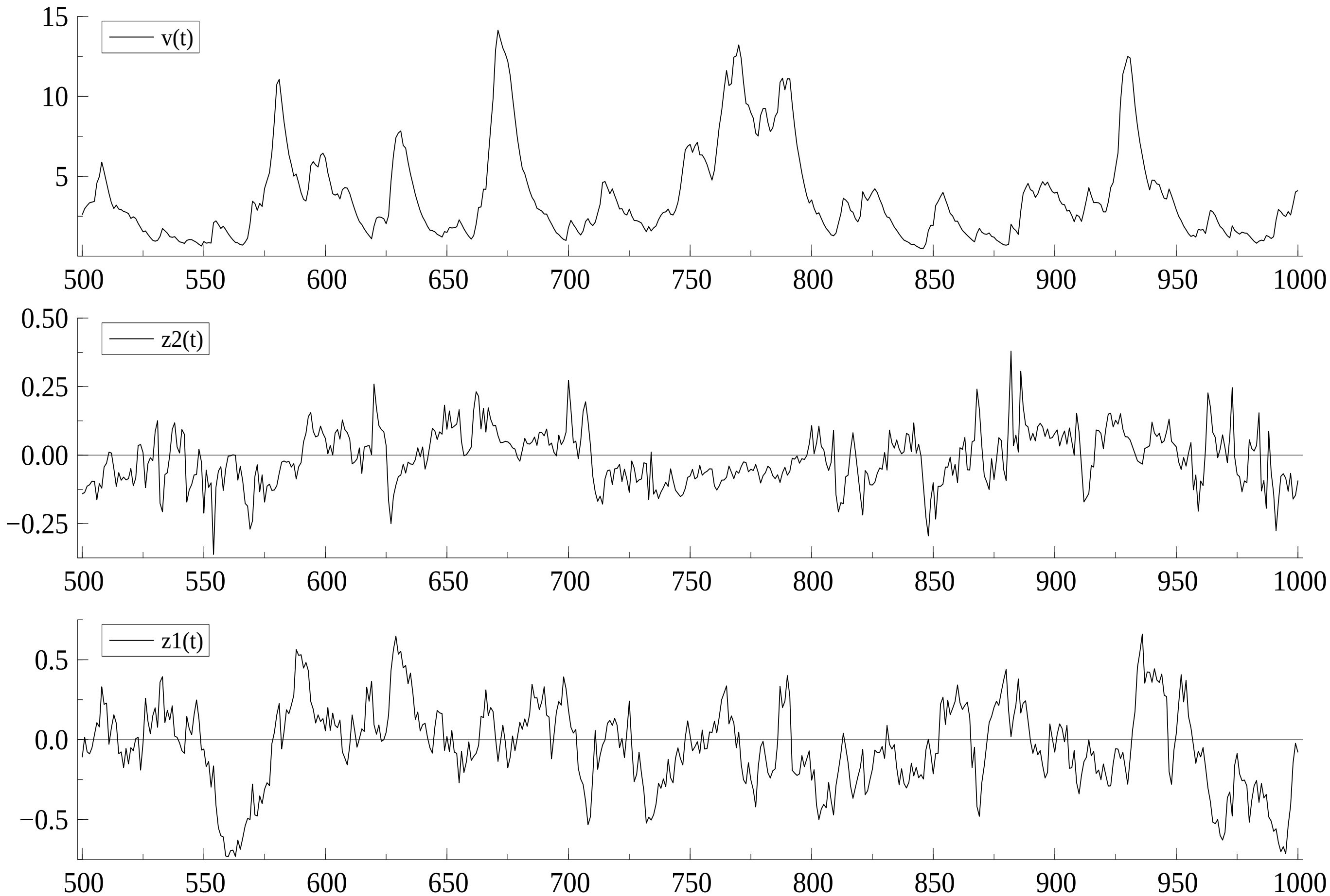


Figure 11: The geometric volatility (v_t, top) and the demands of the trend followers (z_{2t}, middle) and the fundamentalists (z_{1t}, bottom).

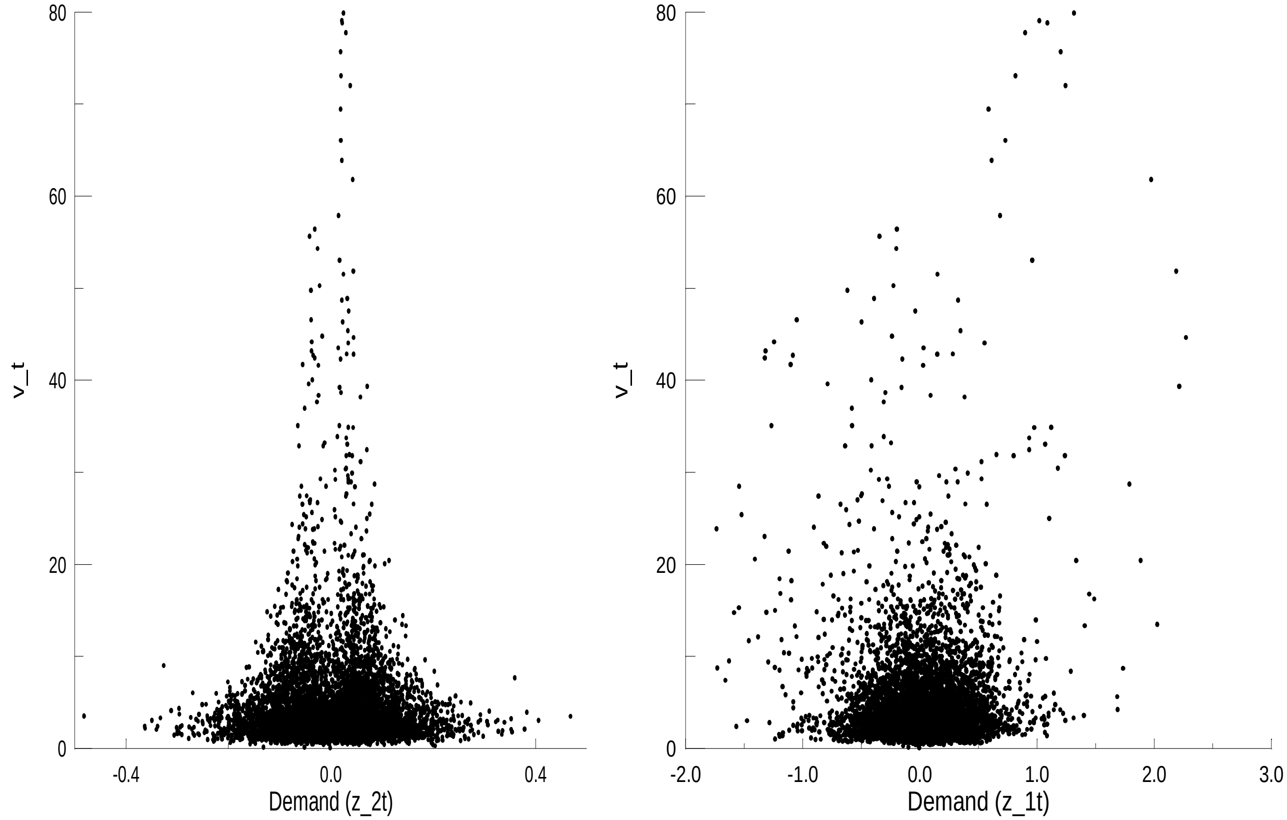


Figure 12: The phase plots of the geometric moving variance (v_t) and the demands of the trend followers (z_{2t}, left) and of the fundamentalists (z_{1t}, right).

- **Overall realistic volatility behaviour is due to**
  - **the interaction of speculators,**
  - **the simple feedback of the trend followers, and**
  - **the interplay of noises and the underlying deterministic dynamics**

6 Empirical Evidence and Long Memory of Financial Indices

6.1 Statistics and Autocorrelations

Table 1: Summary statistics of r_t.

data	mean	std.	skewness	kurtosis	min	max	stud. range	Jarque-Bera
DAX 30	0.0003	0.0143	-0.467	8.940	-0.137	0.076	14.91	7991
FTSE 100	0.0003	0.0105	-0.735	13.07	-0.130	0.076	19.60	22879
NIKKEI 225	0.0000	0.0137	-0.142	10.47	-0.161	0.124	20.78	12365
S&P 500	0.0004	0.0107	-1.997	45.96	-0.228	0.087	29.35	411423

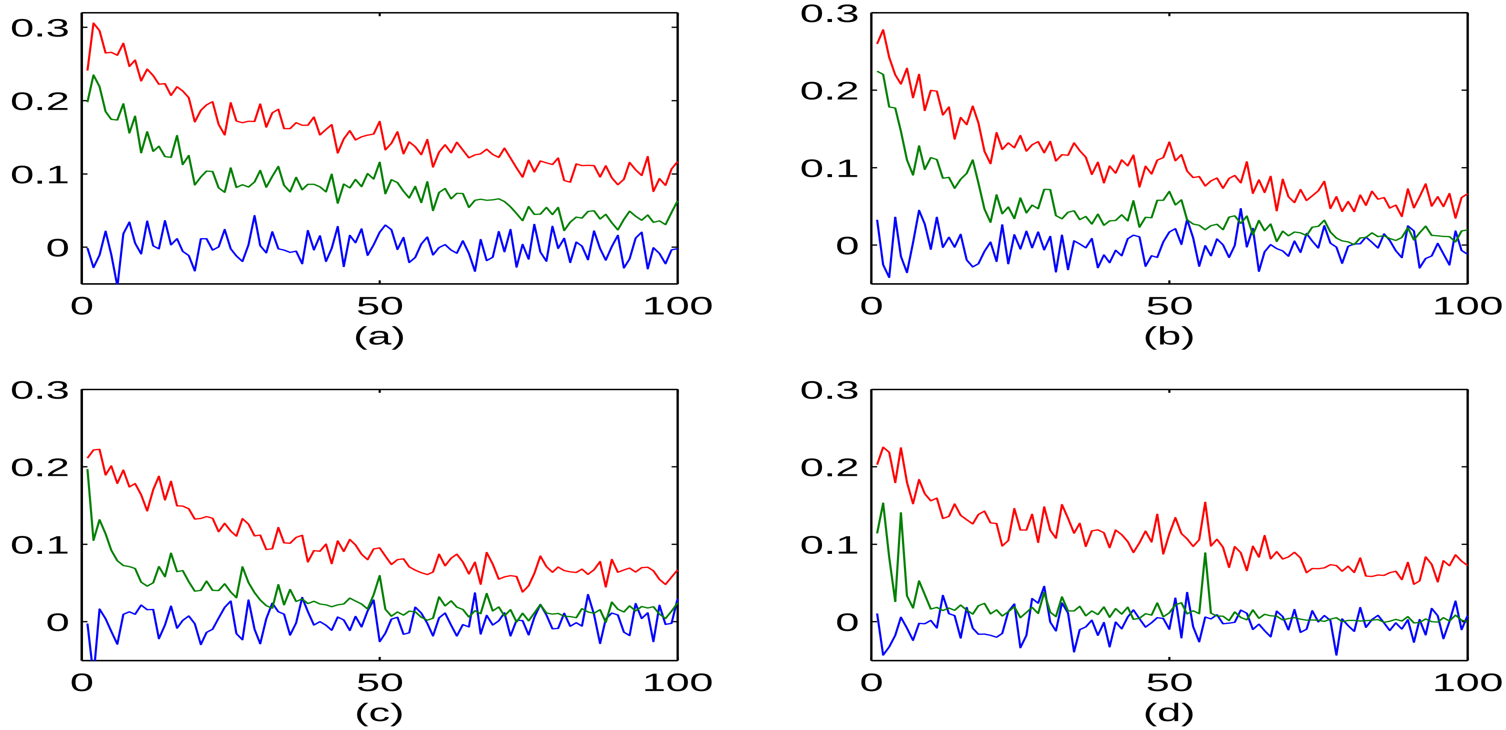


Figure 13: Autocorrelations of returns, the squared returns and the absolute returns for the DAX 30 (a), the FTSE 100 (b), the NIKKEI 225 (c), and the S&P 500 (d). The lines from the bottom to the top are the autocorrelations for returns, the squared returns, and the absolute returns, respectively.

6.2 Volatility Clustering and GARCH Estimates

Table 2: GARCH $(1, 1)$ Parameter Estimates

data	$a \times 10^3$	b	$\alpha_0 \times 10^4$	α_1	β_1
DAX 30	0.655(0.161)	0.0335(0.0162)	0.048(0.004)	0.1185(0.0049)	0.8604(0.0071)
FTSE 100	0.514(0.120)	0.0404(0.0149)	0.023(0.003)	0.0966(0.0066)	0.8824(0.0085)
NIKKEI 225	0.751(0.138)	0.0415(0.0150)	0.023(0.003)	0.1392(0.0036)	0.8608(0.0046)
S&P 500	0.600(0.119)	0.0267(0.0154)	0.013(0.002)	0.0797(0.0020)	0.9114(0.0036)

Note: The numbers in parentheses are standard errors. This also holds for Table 3.

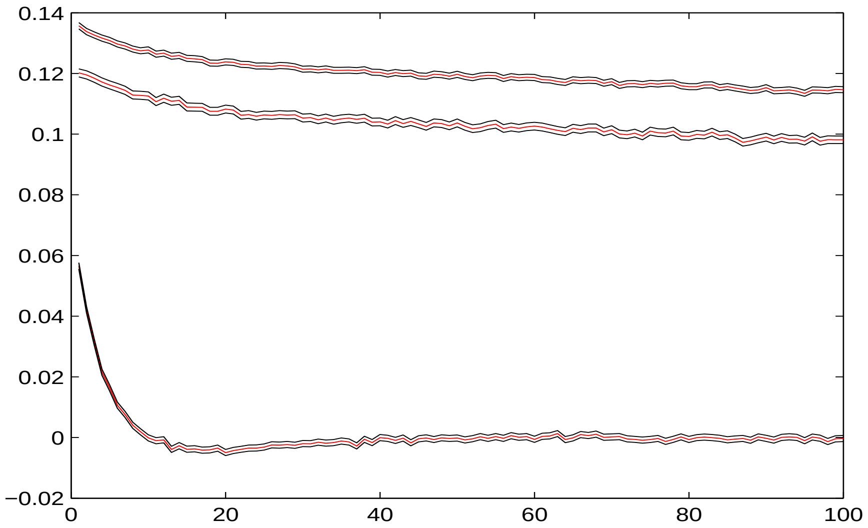
6.3 Long Memory and FIGARCH Model

Table 3: FIGARCH $(1, d, 1)$ Parameter Estimates

data	$a \times 10^3$	b	$\alpha_0 \times 10^4$	d	ϕ_1	β
DAX 30	0.694(0.142)	0.0358(0.0144)	0.933(0.057)	0.0675(0.0129)	0.9608(0.0044)	0.9059(0.008
FTSE 100	0.528(0.118)	0.0459(0.0143)	0.673(0.093)	0.3270(0.0259)	0.0150(0.0556)	0.2559(0.073
NIKKEI 225	20.75(0.070)	-0.0460(0.0010)	0.056(0.024)	0.4047(0.0046)	0.1454(0.0029)	0.7542(0.002
S&P 500	0.629(0.116)	0.0290(0.0158)	0.665(0.094)	0.3353(0.0202)	0.2765(0.0367)	0.5032(0.044

7 Econometric Estimation of the Long Memory of the MF Model

7.1 Autocorrelations



7.2 (FI)GARCH Estimates

Table 4: The GARCH $(1, 1)$ Parameter Estimates for the MF Model

$a \times 10^3$	b	$\alpha_0 \times 10^4$	α_1	β
0.0740	0.0725	0.0078	0.0260	0.9738
(0.2300)	(0.0139)	(0.0035)	(0.0032)	(0.0033)
47	77.1	17.7	100	100

Note: The numbers in parentheses are the standard errors, and the numbers in the last row are the percentages that the test statistics are significant at 5% level over 1000 independent simulations. This also holds for Table 5.

Table 5: The FIGARCH $(1, d, 1)$ Parameter Estimates for the MF Model

a	b	$\alpha_0 \times 10^4$	d	ϕ_1	β
0.0137	0.0769	0.3620	0.3797	0.3439	0.7933
(0.0010)	(0.0195)	(0.6112)	(0.0386)	(0.0281)	(0.0295)
41.2	72.6	35.6	87.6	83.1	98.5

8 Conclusions

- Incorporating a realistic trading period into a simple MF model with heterogeneous beliefs.
- Showing that the long-run behaviour and convergence of the market prices, long (short)-run profitability of the fundamental (trend following) trading strategy, survivability of chartists can be characterized by the dynamics of the underlying deterministic system.
- Linking various under and over-reaction autocorrelation patterns of returns to the bifurcation nature of the underlying deterministic system.
- The simple MF model shows clearly the long memory properties

- **The long memory mechanism is different from either herding or switching mechanisms, but it shares the same spirt in a much simple way**
- **It is this simplicity that make it possible to identify potential source and mechanism to generate certain characteristics and this is one of the contributions of this paper.**

9 Future Work

- **Market mood and switching: both fixed and changing market fractions.**
- **Market equilibrium distribution via Stochastic Bifurcation Method**
- **Model estimations and calibrations**
- **Multi-assets market and intertemporal optimizations.**
- **A unified framework in continuous-time framework**