Lecture Three

Aggregation of Heterogeneous Beliefs and the CAPM under Mean-Variance Framework

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Plan of Talk

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- Other Setups:
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    - Market Set Up and Heterogeneous Beliefs
    - Consensus Belief and Equilibrium CAPM-like Relationship under Heterogeneous Beliefs
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    - Statistical Analysis on the Impact of the Heterogeneity
  - Heterogeneous Beliefs in Payoffs without Risk-free Asset
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  - Heterogeneous Beliefs in Returns without Risk-free Asset
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1 Literature and Motivation

- **Sharpe-Lintner-Mossin Capital Asset Pricing Model (CAPM)**:
  - Plays a central role in finance theory
  - Paradigm of homogeneous beliefs and a rational representative agent.
  - Criticisms from theoretic and empirical points
- **Related Literature**:
  - **Heterogeneous beliefs can affect aggregate market returns**.
  - Typically the **heterogeneous beliefs** reflect
    * Difference of risk attitude—Huang and Litzenberger (1988)
    * Difference of opinion among the agents in complete market—Lintner (1969), Miller (1977), Mayshar (1982), Varian (1985), Abel (1989, 2002), Cecchetti *et al.*(2000).

  - ∗ Difference of information upon which agents are trying to learn—Williams (1977), Detemple and Murthy (1994) and Zapatero (1998).
- **Studies in different framework**:
  - ∗ CAPM-like mean-variance models—Lintner (1969), Miller (1977) Williams (1977) and Mayshar (1982);
  - ∗ Arrow-Debreu economy—Varian (1985), Abel (1989, 2002), Calvet *et al.* (2004) and Jouini and Napp (2006).
- **Studies in different portfolio**:
  - ∗ Portfolio of one risky asset and one risk-free asset—Abel (1989), Zapatero (1998), Basak (2000) and Johnson (2004). Including boundedly rational and heterogeneous agents literature—survey papers by Hommes (2006) and LeBaron (2006).
  - ∗ Portfolio of many risky assets and one risk-free asset—Williams (1977), Varian (1985) and Jouini and Napp (2006).

- **Focus on** the heterogeneous in the risk preferences and expected payoffs or returns of risky assets, rather than the variances and co-variances, except Lintner (1969).

- **Empirical Studies on the divergence of opinion and stock price**:
  - Miller (1977) propose a direct relationship between a stock's *risk* and its *divergence of opinion* and argue that the market clearing price of stocks with divergence of opinion will be higher.
  - Diether et al. (2002) provide an empirical evidence that stocks with higher dispersion in analysts' earnings forecasts earn lower future returns than otherwise similar stocks, in particularly for small stocks and stocks that have performed poorly over the past year.
  - Ang *et al.* (2006) examine relation between cross-sectional volatility and expected returns and find that stocks with high sensitivities to innovations in aggregate volatility have low average returns.

- **Questions**: When the market is characterized by heterogeneous investors with different risk preferences and different beliefs on expected payoff and variance/covariance matrices of stocks' payoffs,
  - how does the market aggregate the heterogeneous beliefs and determine the market clearing prices?
  - what are the impact of heterogeneous beliefs on the market equilibrium price, returns and βs of stocks?
  - is heterogeneity good or bad for the market in general?
  - can the heterogeneity be used to explain some stylized facts, including non-normality of return distribution, equity risk premium and risk-free rate puzzles, and cross-sectional returns?

- **Plan of this lecture**:
  - to consider two different setups, payoffs and returns, and two situations with and without risk-free asset;
  - to introduce heterogeneous beliefs in risk preferences, means and variances/covariances among agents within the mean-variance framework;
  - to analyze the aggregation properties of their heterogeneous beliefs;
  - to establish (zero-beta) CAPM-like relationships under heterogeneous beliefs;
  - to examine the impact of the heterogeneity on asset equilibrium price.

2 Mean-Variance Analysis under Heterogeneous Beliefs in Payoffs with a Risk-free Asset

- **A static mean-variance model** by allowing the agents to have distinct subjective means, variances and covariances.
- **Market**:
  - one risk-free asset with payoff $R_f = 1 + r_f$;
  - $K (\geq 1)$ risky assets with payoff:

$$\tilde{\mathrm{x}} = (\tilde{x}_1, \cdots, \tilde{x}_K)^T, \qquad \tilde{x}_k = \tilde{p}_k + \tilde{d}_k.$$

- **Heterogeneity**: There are I investors. The heterogeneous (subjective) belief $\mathcal{B}_i = (\mathbb{E}_i(\tilde{\mathrm{x}}), \Omega_i)$ of investor i is defined

$$\mathrm{y}_i = \mathbb{E}_i(\tilde{\mathrm{x}}) = (y_{i,1}, y_{i,2}, \cdots, y_{i,K})^T, \qquad \Omega_i = (\sigma_{i,kl})_{K \times K}.$$

where

$$y_{i,k} = E_i[\tilde{x}_k], \quad \sigma_{i,kl} = Cov_i(\tilde{x}_k, \tilde{x}_l).$$

- **Portfolio and Endowment**: Let $z_{i,o}$ ($\bar{z}_{i,o}$) be the amount (endowment) of investor i in the risk-free asset, and

  $$\mathbf{z}_i = (z_{i,1}, z_{i,2}, \cdots, z_{i,K})^T \quad \text{and} \quad \bar{\mathbf{z}}_i = (\bar{z}_{i,1}, \bar{z}_{i,2}, \cdots, \bar{z}_{i,K})^T$$

  be the risky portfolio/ endowment in absolute amount of the risky assets.

- **Portfolio Wealth** for investor i:

  $$\tilde{W}_i = R_f z_{io} + \tilde{\mathbf{x}}^T \mathbf{z}_i.$$

- **The mean and variance**:

$$\mathbb{E}_i(\tilde{W}_i) = x_o z_{i,o} + \mathbf{y}_i^T \mathbf{z}_i, \qquad \sigma_i^2(\tilde{W}_i) = \mathbf{z}_i^T \Omega_i \mathbf{z}_i.$$

- **Assumptions**:

(H1) Assume the expected utility of the wealth generated from the portfolio $(z_{i,o}, \mathbf{z}_i)$ of investor i has the form $V_i(\mathbb{E}_i(\tilde{W}_i), \sigma_i^2(\tilde{W}_i))$, where $V_i(x, y)$ is continuously differentiable and satisfies

$$V_{i1}(x, y) = \frac{\partial V_i(x, y}{\partial x} > 0, \qquad V_{i2}(x, y) = \frac{\partial V_i(x, y)}{\partial y} < 0.$$

(H2) Assume

$$\theta_i = -2V_{i2}(x, y)/V_{i1}(x, y) = const.$$

- **Consistent with CARA utility function** $U_i(w) = -e^{-A_i w}$ with $\theta_i = A_i$.

- **Portfolio Maximization Problem**:

$$\max_{z_{i,o}, \mathbf{z}_i} V_i(\mathbb{E}_i(\tilde{W}_i), \sigma_i^2(\tilde{W}_i))$$

subject to the budget constraint

$$z_{i,o} + \mathbf{p}_o^T \mathbf{z}_i = \bar{z}_{i,o} + \mathbf{p}_o^T \bar{\mathbf{z}}_i.$$

- **Lemma—Optimal Portfolio of Heterogeneous Agent**: Under assumptions (H1) and (H2), the optimal risky portfolio $\mathbf{z}_i^*$ of investor i at the market equilibrium is given by

$$\mathbf{z}_i^* = \theta^{-1}\Omega_i^{-1}[\mathbf{y}_i - R_f \mathbf{p}_o]. \tag{1}$$

3 Consensus Belief and Equilibrium Asset Pricing

- **Market aggregate condition and market portfolio**:

$$\sum_{i=1}^{I} \mathbf{z}_i^* = \sum_{i=1}^{I} \bar{\mathbf{z}}_i := z_m \qquad (2)$$

- **A market equilibrium**: a vector of asset prices $\mathbf{p}_o$ determined by (1) together with the market aggregate condition (2).
- **Consensus belief**: A belief $\mathcal{B}_a = (\mathbb{E}_a(\tilde{\mathrm{x}}), \Omega_a)$ is called a **consensus belief** iff the equilibrium price under the heterogeneous beliefs is also the equilibrium price under homogeneous belief $\mathcal{B}_a$.

- **Proposition 1**: Under assumptions (H1) and (H2), let

$$\Theta = \left[\frac{1}{I}\sum_{i=1}^{I}(1/\theta_i)\right]^{-1}$$

Then

(i) the consensus belief $\mathcal{B}_a$ is defined by

$$\Omega_a = \Theta^{-1}\left(\frac{1}{I}\sum_{i=1}^{I}\theta_i^{-1}\Omega_i^{-1}\right)^{-1}$$

$$\mathbb{E}_a(\tilde{\mathbf{x}}) = \Theta\Omega_a\left(\frac{1}{I}\sum_{i=1}^{I}\theta_i^{-1}\Omega_i^{-1}\mathbb{E}_i(\tilde{\mathbf{x}}_i)\right)$$

(ii) the market equilibrium price $\mathbf{p}_o$ is determined by

$$\mathbf{p}_o = \frac{1}{R_f}\left[\mathbb{E}_a(\tilde{\mathbf{x}}) - \frac{1}{I}\Theta\Omega_a\mathbf{z}_m\right]$$

(iii) the equilibrium optimal portfolio of agent i is given by

$$\mathbf{z}_i^* = \theta_i^{-1}\Omega_i^{-1}\left[(\mathbf{y}_i - \mathbf{y}_a) + \frac{1}{I}\Theta\Omega_a\mathbf{z}_m\right]$$

4 Aggregation Properties and Impact of Heterogeneity

4.1 Aggregation effect of diversity in risk aversion coefficients

- Θ: the **harmonic mean** of the absolute risk aversions of all investors.
- **Diversification Effect**:

$$\Theta < \frac{1}{I}\sum_i \theta_i$$

- **The mean-preserving spread (MPS) effect**:
  - MPS was developed in Rothschild-Stiglitz (1970) to measure the stochastic dominance among risky assets.
  - For $I = 2$, assume investor-2 is more risk averse than investor-1:

$\{\theta_1, \theta_2\}$ with $\theta_1 < \theta_2$.

– Let $\overline{\theta} := (\theta_1 + \theta_2)/2$ be the mean (or average) risk aversion. The aggregate risk aversion in this case can be written as follows

$$\Theta = 2\frac{\theta_1\theta_2}{\theta_1 + \theta_2} = \frac{\theta_1\theta_2}{\overline{\theta}}$$

– Consider an MPS in the risk aversion coefficients

$$\{\theta_1', \theta_2'\} = \{\theta_1 - \varepsilon, \theta_2 + \varepsilon\}$$

in which $\theta_1 \geq \varepsilon > 0$ measure the dispersion of heterogeneous belief in the risk aversions around the mean.

– Then $\Theta' < \Theta$, meaning diversity of an MPS in risk-aversion coefficients can reduce the aggregate risk aversion coefficient.

– In particular, if $\epsilon = \theta_1$, the aggregate risk aversion $\Theta' = 0$, implying a risk-neutral market.

- **Conclusion**: Aggregation of diversified risk preferences among heterogeneous agents make the market become less risk averse, even risk neutral in some situation.

- Other Implications: In this case, $\boldsymbol{\Omega}_a = \boldsymbol{\Omega}_i = \boldsymbol{\Omega}_o$ and $\mathbb{E}_a(\tilde{\mathbf{x}}) = \mathbb{E}_i(\tilde{\mathbf{x}}) = \mathbb{E}_o(\tilde{\mathbf{x}})$. Then

$$\mathbf{p}_o = \frac{1}{R_f}\left[\mathbb{E}_o(\tilde{\mathbf{x}}) - \frac{1}{I}\boldsymbol{\Theta}\boldsymbol{\Omega}_o\mathbf{z}_m\right], \qquad \mathbf{z}_i^* = \frac{\boldsymbol{\Theta}}{I\theta_i}\mathbf{z}_m$$

  - The standard one-fund theorem holds.
  - For a given $\boldsymbol{\Theta}$, the market is dominated by less-risk averse investors.
  - A lower aggregate risk aversion coefficient $\boldsymbol{\Theta}$ (due to diversified beliefs) leads to higher market equilibrium prices and lower expected market returns.
  - When $\boldsymbol{\Theta} \approx \mathbf{0}$ (a risk-neutral market), the market expected returns of the risky assets are close to the risk-free return.

4.2 Aggregation effect of diversity in variances and covariances

- If $\mathbb{E}_i(\tilde{\mathbf{x}}) = \mathbb{E}_o(\tilde{\mathbf{x}})$, then $\mathbb{E}_a(\tilde{\mathbf{x}}) = \mathbb{E}_o(\tilde{\mathbf{x}})$ and

$$\mathbf{p}_o = \frac{1}{R_f}\left[\mathbb{E}_o(\tilde{\mathbf{x}}) - \frac{1}{I}\boldsymbol{\Theta}\boldsymbol{\Omega}_a \mathbf{z}_m\right], \qquad \mathbf{z}_i^* = \frac{\boldsymbol{\Theta}}{I\theta_i}\boldsymbol{\Omega}_i^{-1}\boldsymbol{\Omega}_a \mathbf{z}_m$$

The inverse of **aggregate covariance** is a weighted average of the inverse of the risk-adjusted **heterogeneous covariances**.

- One-fund theorem does not hold.
- **Two questions**:
  - Q1: Does the market aggregation generate lower than the average risk for any portfolio?

– Q2: What is the market role on the risk of the asset with more diversified covariance beliefs?

- Answer to Q1:
  – We would like to see if $\boldsymbol{\sigma_a^2(\mathbf{z}) \le \overline{\sigma^2}(\mathbf{z})}$, where

$$\boldsymbol{\sigma_a^2(\mathbf{z}) = \mathbf{z}^T\Omega_a\mathbf{z}, \quad \overline{\sigma^2}(\mathbf{z}) = \mathbf{z}^T\bar{\Omega}\mathbf{z}, \quad \bar{\Omega} = (\Theta/I)\sum_{i=1}^{n}\theta_i^{-1}\Omega_i.}$$

  – **A special case**: Assume the payoffs of all risky assets are **uncorrelated**, then

$$\boldsymbol{(\sigma_{a,j}^2)^{-1} = \frac{\Theta}{I}\sum_{i=1}^{I}\theta_i^{-1}(\sigma_{i,j}^2)^{-1},}$$

and hence

$$\sigma_{a,j}^2 < \bar{\sigma}_{a,j}^2 = \frac{\Theta}{I}\sum_{i=1}^{I}\theta_i^{-1}\sigma_{i,j}^2.$$

- Hence the variance of any portfolio under the aggregate variance is smaller than that under the weighted average variance—**variance diversification under heterogeneous beliefs**.

- Answer to Q2:
  - **MPS in variances**:
    - If $\theta_i = \theta$, an MPS in variance beliefs can reduce the asset risk under the aggregation.
    - However, this result is true under ceratin condition when $\theta_i \neq \theta$.
  - **Example**:
    - an MPS in variance beliefs reduces asset's risk when investor who believes the asset is more risky is more risky averse.

    * an MPS in variance beliefs increases asset's risk when investor who believes the asset is more risky is less risky averse.

- **Consistent with Miller's proposition and the empirical evidence**:
  - By assuming that investors are risk averse, we can argue that investors who believe an asset is more risky are more risk averse.
  - Assume the investors have homogeneous beliefs in expected payoffs but heterogeneous in risk aversion coefficients and variances of the assets.
  - The result can be used to explain the empirical relation between cross-sectional volatility and expected returns— stocks with higher dispersion in analysts' earnings forecasts earn lower future returns than otherwise similar stocks.
  - In other word, stocks with higher dispersion in expected payoffs have higher market clearing prices and earn lower future expected returns than otherwise similar stocks.

– This kind of argument cannot be hold when investors have homogeneous belief.

4.3 Aggregation effect of diversity in expected payoffs

- If $\mathbb{E}_i(\tilde{\mathbf{x}}) \neq \mathbb{E}_j(\tilde{\mathbf{x}})$, then one-fund theorem does not hold in general.
- If $\mathbb{E}_i(\tilde{\mathbf{x}}) = \mathbb{E}_o(\tilde{\mathbf{x}})$, then $\mathbb{E}_a(\tilde{\mathbf{x}}) = \mathbb{E}_o(\tilde{\mathbf{x}})$, although investors may disagree on their risk preferences, variances and covariances.
- If investors agree on the variance and covariance $\mathbf{\Omega}_i = \mathbf{\Omega}$, then

$$\mathbb{E}_a(\tilde{\mathbf{x}}) = \frac{1}{I}\sum_{i=1}^{I}\frac{\Theta}{\theta_i}\mathbb{E}_i(\tilde{\mathbf{x}}_i); \qquad \mathbf{p}_o = \frac{1}{R_f}\left[\mathbb{E}_a(\tilde{\mathbf{x}}) - \frac{1}{I}\Theta\mathbf{\Omega}\mathbf{z}_m\right]$$

reflecting a **weighted average opinion** of the market.

- The market is dominated by investors who are less (more) risk averse and believe higher (lower) expected payoff, as what we would expect in bull (bear) market.
- The market may be unchanged even if investors have divergent opinions on their expected payoffs, as long as they are *balanced*.

- The market expected payoff $\mathbb{E}_{\boldsymbol{a}}(\tilde{\mathbf{x}})$ is affected by the covariance beliefs only when investors disagree on both the expected payoffs and covariances.

4.4 Impact on the market equilibrium price

- The equilibrium price formula is exactly the same as the traditional equilibrium price for a representative agent holding the consensus belief $\mathcal{B}_a$.
- **Price Aggregation**:
  - If $\mathbf{p}_{i,o}$ is the equilibrium price for investor i as if he/she was the only investor in the market, then then we would have

$$\mathbf{p}_{i,o} = \frac{1}{R_f}[\mathbb{E}_i(\tilde{\mathbf{x}}) - \theta_i \Omega_i \bar{z}_i].$$

  - Hence the market equilibrium price is a weighted average of each agent's equilibrium prices under his/her belief.

$$\mathbf{p}_o = \Theta \Omega_a \left[\frac{1}{I} \sum_{i=1}^{I} \theta_i^{-1} \Omega_i^{-1} \mathbf{p}_{i,o} \right]. \tag{3}$$

– Consistent with Miller's argument, the market price may reflect the expectations of only the most optimistic minority, as long as this minority can absorb the entire supply of stock.

- **The equity risk premium** (ERP)— $\Theta\Omega_a \mathbf{z}_m / I$:
  – An MPS variance belief when asset payoffs are uncorrelated will reduces the aggregated variances of stocks, leading to a lower ERP and therefore a higher market price, lower expected return.
  – ERP becomes smaller when the number of investors increases. In the limiting case, $ERP \longrightarrow 0$ as $I \longrightarrow \infty$, leading to the traditional *risk-neutral discount equity value formula*

$$\mathbf{p}_o \approx \frac{1}{R_f}\mathbb{E}_a(\tilde{\mathbf{x}}) = \frac{1}{R_f}\mathbb{E}_a(\tilde{\mathbf{p}} + \tilde{\mathbf{d}});$$

4.5 Impact on the optimal demands and trading volume

- The equilibrium demand of individual investor depends on both
  - the **dispersion** of his/her expected payoff from the aggregate average expected payoff $\mathbb{E}_i(\tilde{\mathbf{x}}) - \mathbb{E}_a(\tilde{\mathbf{x}})$ and
  - the covariance of risky asset and the market portfolio.
- For a special case when $\theta_i = \theta_o, \Omega_i = \Omega_o$,

$$\mathbf{z}_i^* = \theta_o^{-1}\Omega_o^{-1}[\mathbb{E}_i(\tilde{\mathbf{x}}) - \mathbb{E}_a(\tilde{\mathbf{x}})] + \mathbf{z}_m/I,$$

the market equilibrium price is reduced to

$$\mathbf{p}_o = \frac{1}{R_f}[\mathbb{E}_a(\tilde{\mathbf{x}}) - \theta_o\Omega_o\mathbf{z}_m/I], \qquad \mathbb{E}_a(\tilde{\mathbf{x}}) = \frac{1}{I}\sum_{i=1}^{I}\mathbb{E}_i(\tilde{\mathbf{x}}).$$

- Price and trading volume may or may not be related to each other:

– An MPS in the distribution of the expected payoffs among investors will not change the equilibrium price, but will increase the trading volume in the market;

– A higher (or lower) market price due to a higher (or lower) averaged expected payoff may not necessarily lead to high trading volume.

5 The CAPM-like Relationship under Heterogeneous Beliefs

- **CAPM-like price relation**: HCAPM

$$\mathbb{E}_a(\tilde{\mathbf{x}}) - R_f \mathbf{p}_o = \frac{1}{\sigma_m^2}\Omega_a \mathbf{z}_m[\mathbb{E}_a(\tilde{W}_m) - R_f W_{m,o}],$$

or equivalently,

$$\mathbb{E}_a(\tilde{x}_k) - R_f p_{k,o} = \frac{\sigma(\tilde{W}_m, \tilde{x}_k)}{\sigma_m^2}[\mathbb{E}_a(\tilde{W}_m) - R_f W_{m,o}],$$

where $\sigma(\tilde{W}_m, \tilde{x}_k) = \sum_{j=1}^{K} z_{m,j}\sigma_{kj}$ is the payoff covariance of the risky asset k and the market portfolio.

- **CAPM-like return relation**:
  - Let

$$\tilde{r}_j = \frac{\tilde{x}_j}{p_{j,o}} - 1, \quad \tilde{r}_m = \frac{\tilde{W}_m}{W_{m,o}} - 1,$$

$$\mathbf{E}_a(\tilde{r}_j) = \frac{\mathbf{E}_a(\tilde{x}_j)}{p_{j,o}} - 1, \quad \mathbf{E}_a(\tilde{r}_m) = \frac{\mathbf{E}_a(\tilde{W}_m)}{W_{m,o}} - 1.$$

  - The CAPM-like return relation under heterogeneous beliefs:

$$\mathbb{E}_a[\tilde{\mathbf{r}}] - r_f \mathbf{1} = \beta[\mathbb{E}_a(r_m) - r_f],$$

where

$$\beta_k = \frac{W_{m,o}}{p_{k,o}} \frac{\sigma(\tilde{W}_m, \tilde{x}_k)}{\sigma_m^2} = \frac{cov_a(\tilde{r}_m, \tilde{r}_k)}{\sigma_a^2(\tilde{r}_m)}.$$

6 Mean-Variance Efficiency—Geometric Relation

- The one-fund theorem and mean-variance efficiency in the standard mean-variance framework with homogeneous beliefs with a risk-less asset.
- These implications no longer true in general under heterogeneous beliefs.
- Examples with different degrees of heterogeneity: three risky assets, one risk-free asset, and two investors.

6.1 Effect of heterogeneous expected payoffs

- **Example 1** Assume the two investors have the same covariance matrix $\Omega_2 = \Omega_1 = \Omega_o$ and different expected payoffs $\mathbf{y}_1, \mathbf{y}_2$ and absolute risk aversion (ARA) coefficients $(\theta_1, \theta_2) = (3, 3), (4, 2)$ and $(2, 4)$

with $\mathbf{y_2} = \mathbf{y_1} + \mathbf{31}$,

$$\mathbf{y_1} = \begin{pmatrix} \mathbf{6.5974} \\ \mathbf{9.3484} \\ \mathbf{9.7801} \end{pmatrix}, \quad \mathbf{\Omega_o} = \begin{pmatrix} \mathbf{0.6292} & \mathbf{0.1553} & \mathbf{0.2262} \\ \mathbf{0.1553} & \mathbf{0.7692} & \mathbf{0.1492} \\ \mathbf{0.2262} & \mathbf{0.1492} & \mathbf{2.1381} \end{pmatrix}.$$

Hence $\mathbf{y_2} > \mathbf{y_1}$.

- Implications:
  - The standard one fund theorem under homogeneous belief does not held in general, the market portfolio is always efficient, but the optimal portfolios $\mathbf{z}_i^*$ of the investors under the market belief become less efficient.
  - The mean-variance frontier under the market belief is located in between the individual frontiers, with the optimistic investor's frontier having the highest slope.
  - The optimal portfolios of the investors are very close to the market

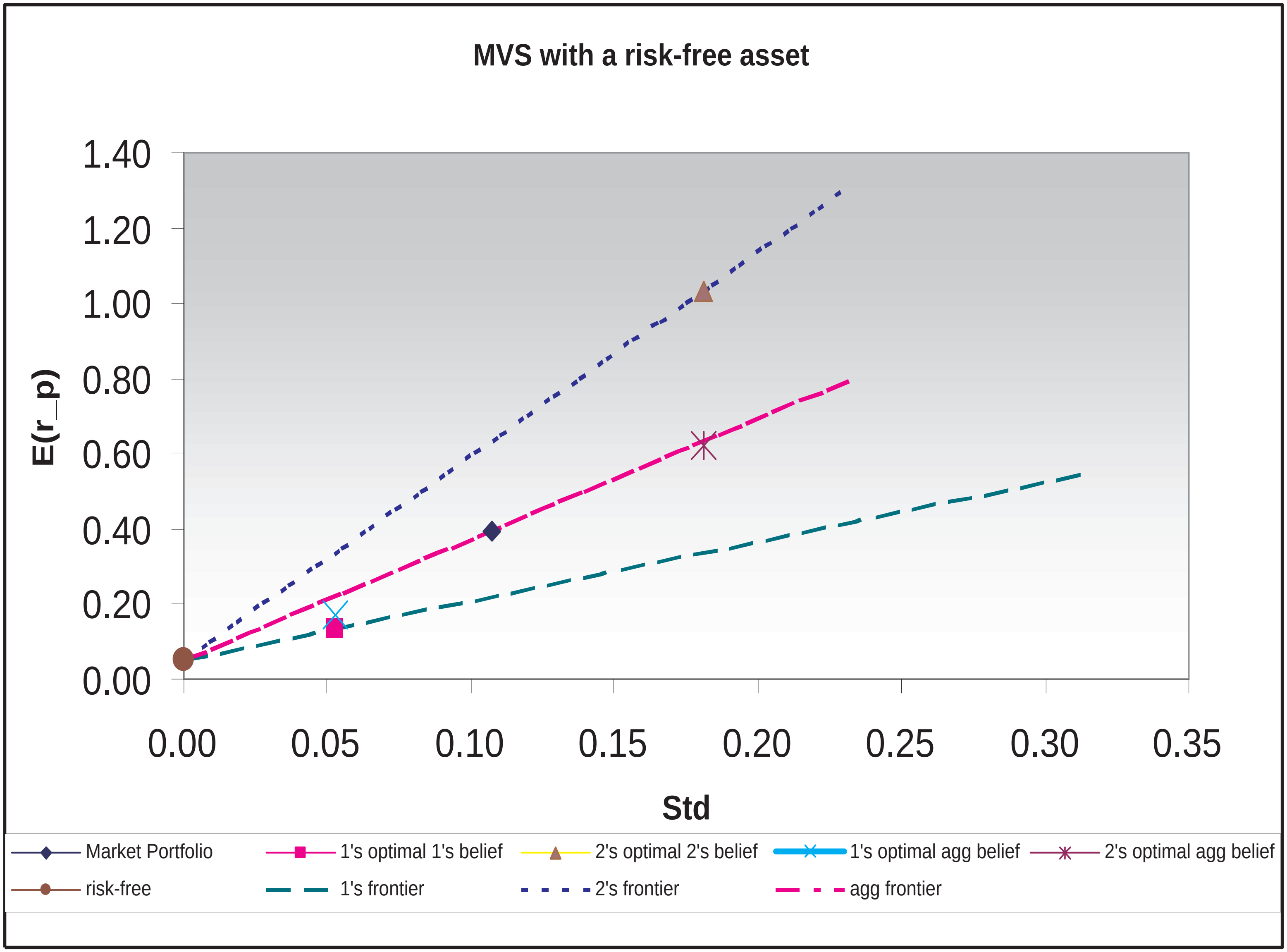
frontier.

– The expected asset payoffs determine the main structure of the diagram while ARAs play a secondary role in the placement of individual optimal portfolios and change of slope of the market's frontier.

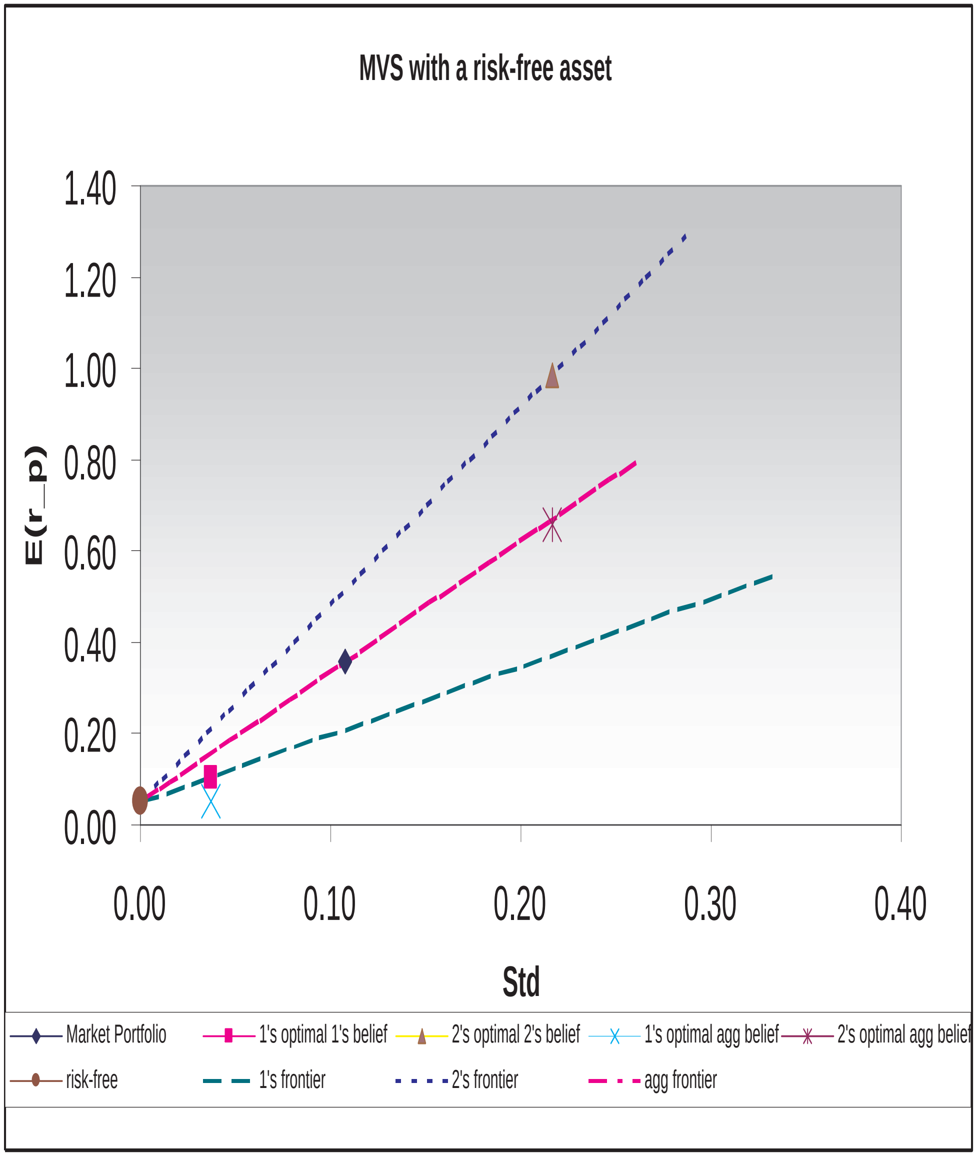
– Under the market equilibrium, the individual's optimal portfolio becomes mean-variance inefficient (efficient) for investor who is optimistic (pessimistic) in the sense that

$$\mu^a_{\mathbf{z}_2^*} < \mu_{\mathbf{z}_2^*}, \quad \mu^a_{\mathbf{z}_1^*} > \mu_{\mathbf{z}_1^*}, \qquad \sigma^a_{\mathbf{z}_2^*} = \sigma_{\mathbf{z}_2^*}, \quad \sigma^a_{\mathbf{z}_1^*} = \sigma_{\mathbf{z}_1^*}.$$

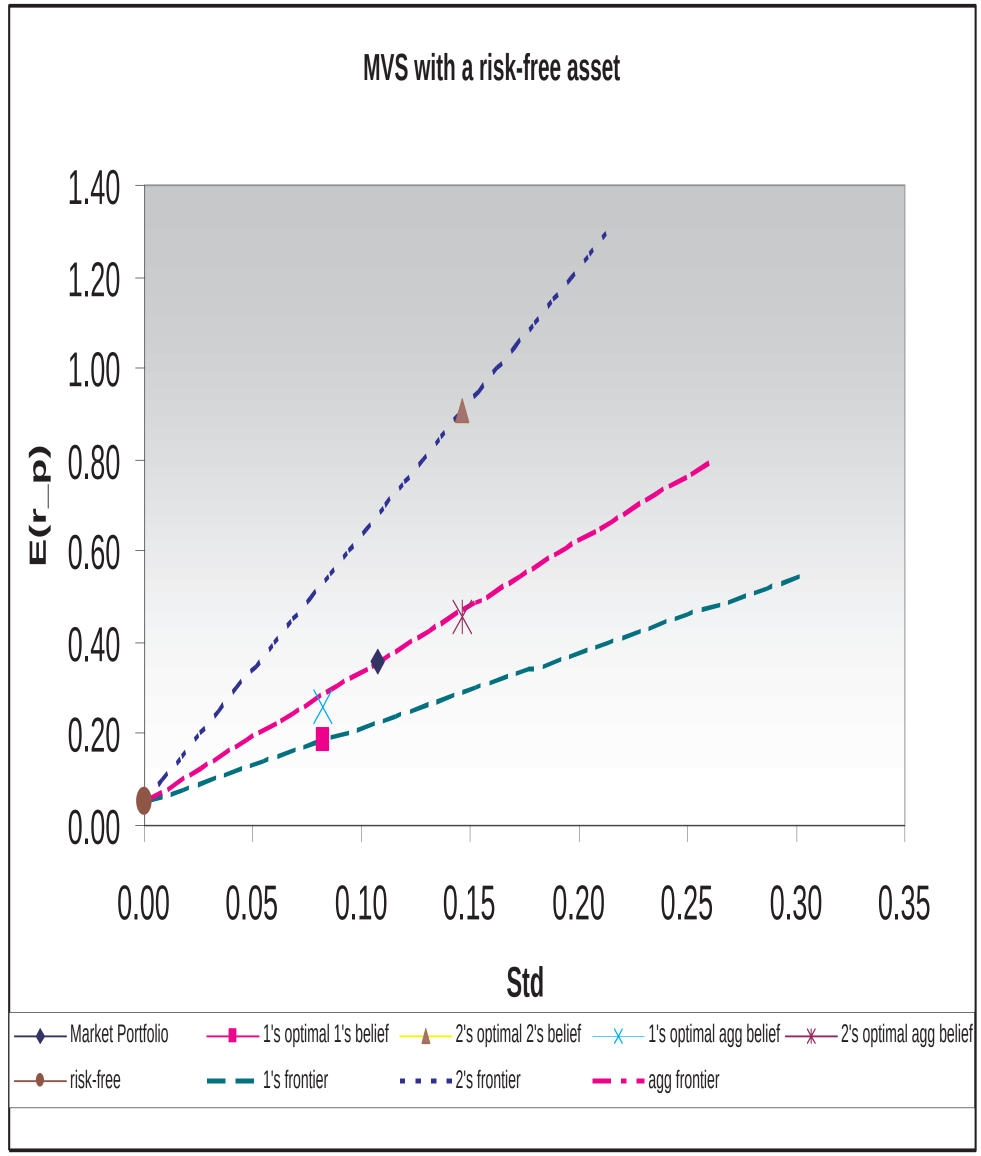
This implies that optimistic investor with respect to the expected payoffs is worsen off under the market equilibrium.



(a1) $(\boldsymbol{\theta_1}, \boldsymbol{\theta_2}) = (\mathbf{3}, \mathbf{3})$



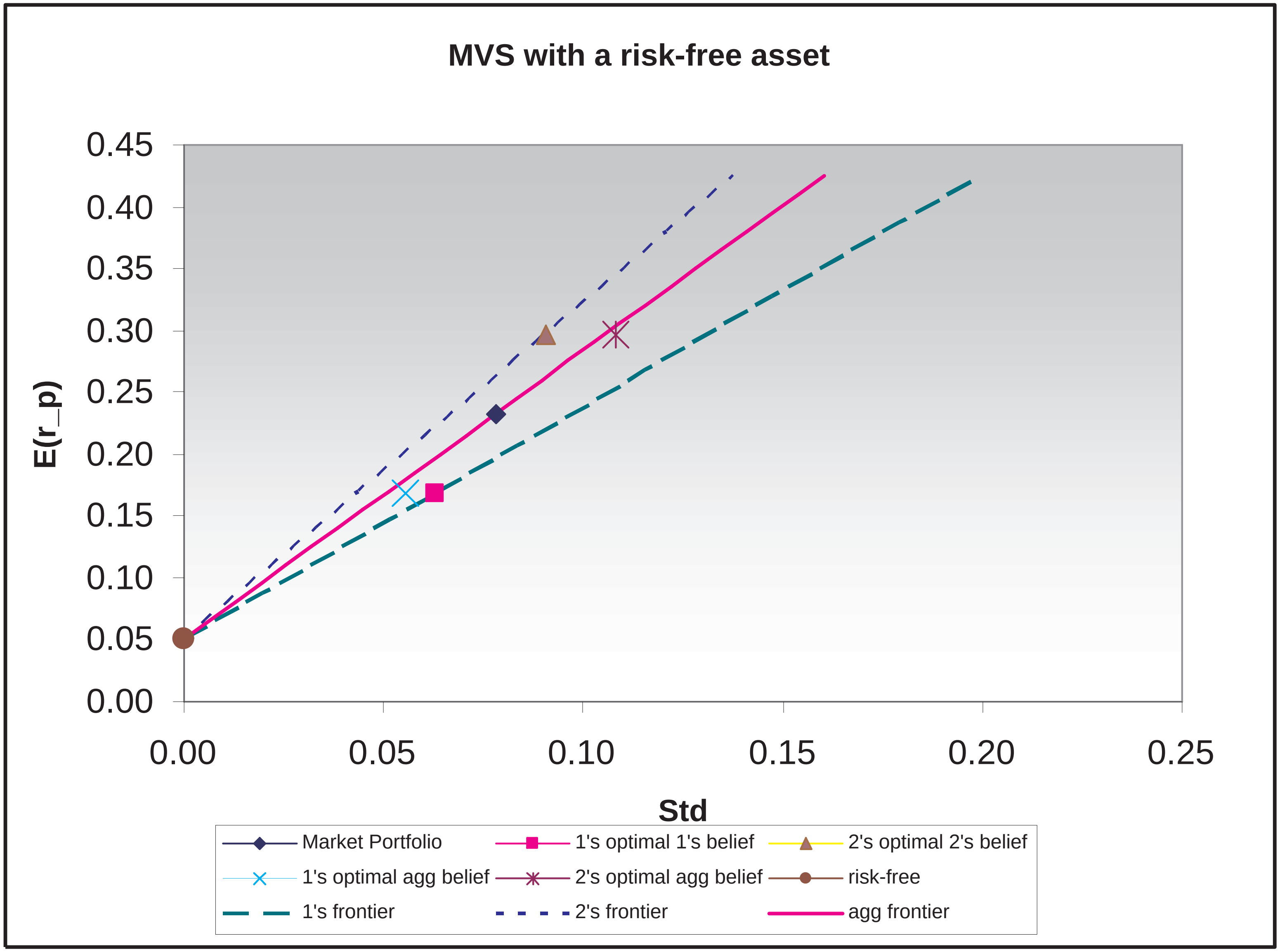
(a2) $(\theta_1, \theta_2) = (4, 2)$



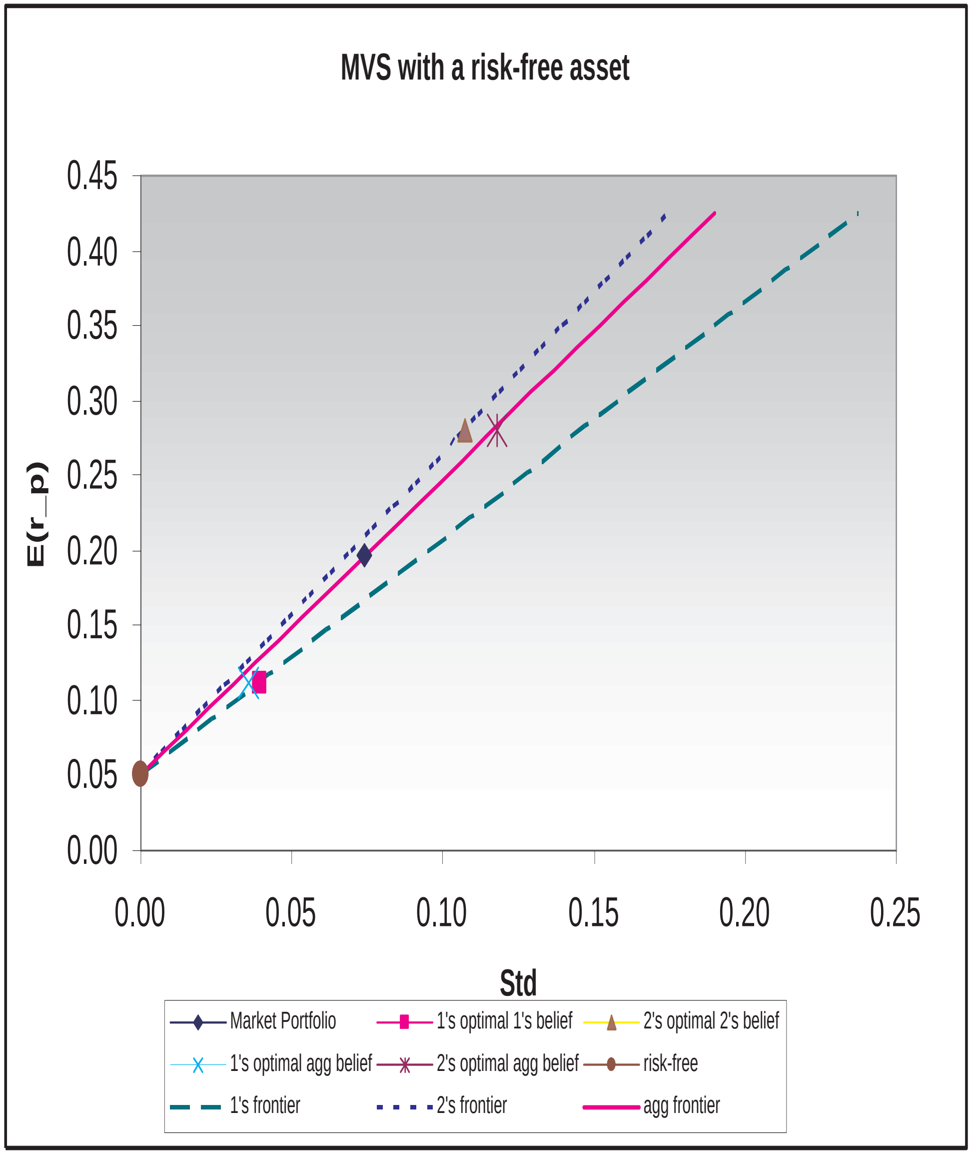
(a3) $(\theta_1, \theta_2) = (2, 4)$

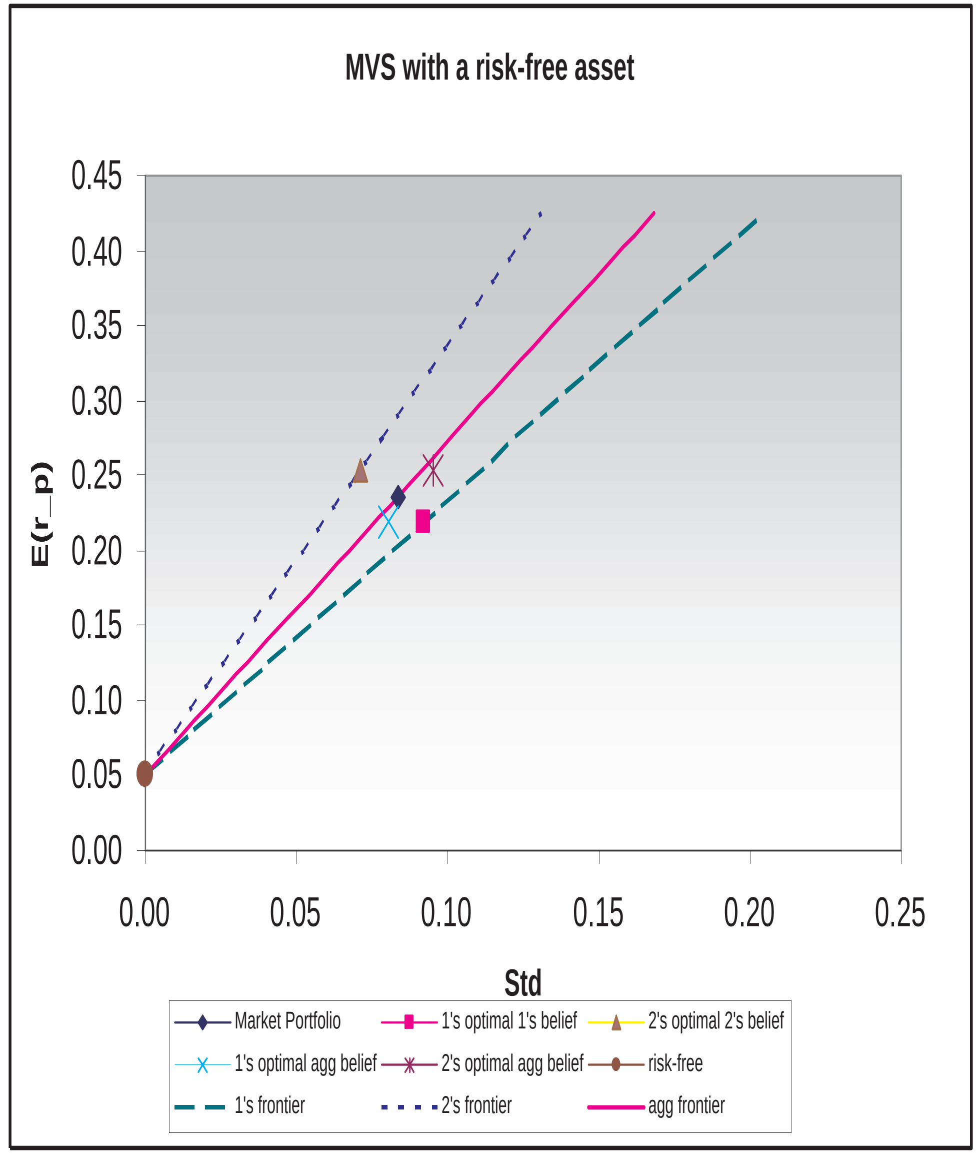
6.2 Effect of heterogeneous variance/covariance matrices

- **Example 2** Let $\mathbf{y_1} = \mathbf{y_2} = \mathbf{y}_o$ and $\mathbf{\Omega_1}$ be the $\mathbf{y_1}$ and $\mathbf{\Omega}_o$ in Example 1, respectively. Let $\mathbf{\Omega_2} = \mathbf{\Omega_1} - \mathbf{0.3} \times \mathbf{1}$. Then $\mathbf{\Omega_1} - \mathbf{\Omega_2}$ is semi-positive definite. Define $\mathbf{\Omega_1} \geq \mathbf{\Omega_2}$ if $\mathbf{\Omega_1} - \mathbf{\Omega_2}$ is semi-positive definite.
- Implications:
  - The standard one fund theorem does not held and the optimal portfolios become inefficient.
  - If we interpret covariance matrix as a risk measure in the sense that investor 2 is more optimistic than investor 1 when $\mathbf{\Omega_2} < \mathbf{\Omega_1}$ and $\mathbf{y_2} = \mathbf{y_1}$. Then the most features in the previous case are still hold.
  - The main structure of the diagram is still determined by the covariance matrices of individuals while the ARAs determine the positions of individuals' optimal portfolios both under their own beliefs and under the consensus belief.



(b1) $(\theta_1, \theta_2) = (3, 3)$



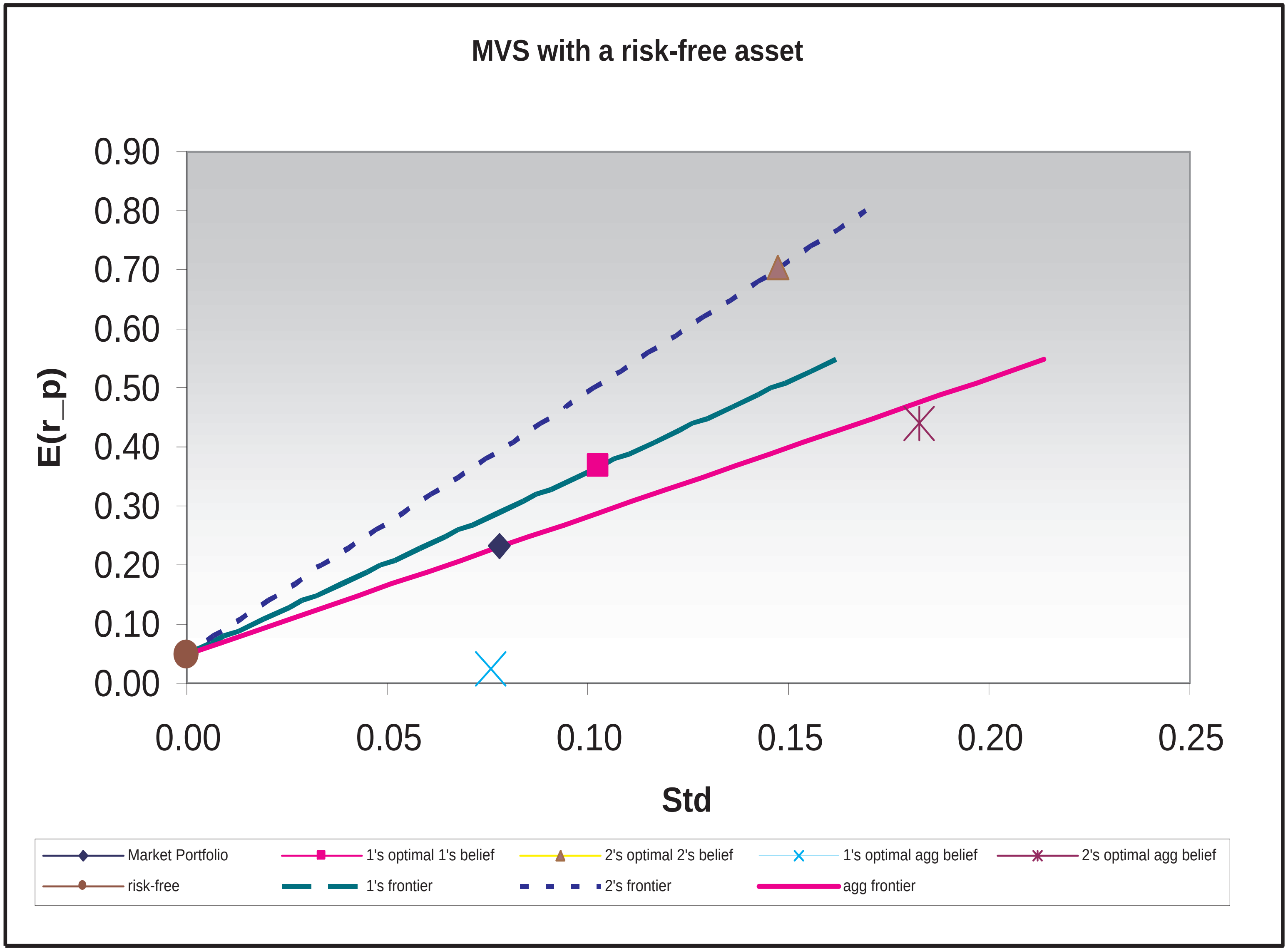


(b2) $(\theta_1, \theta_2) = (4, 2)$

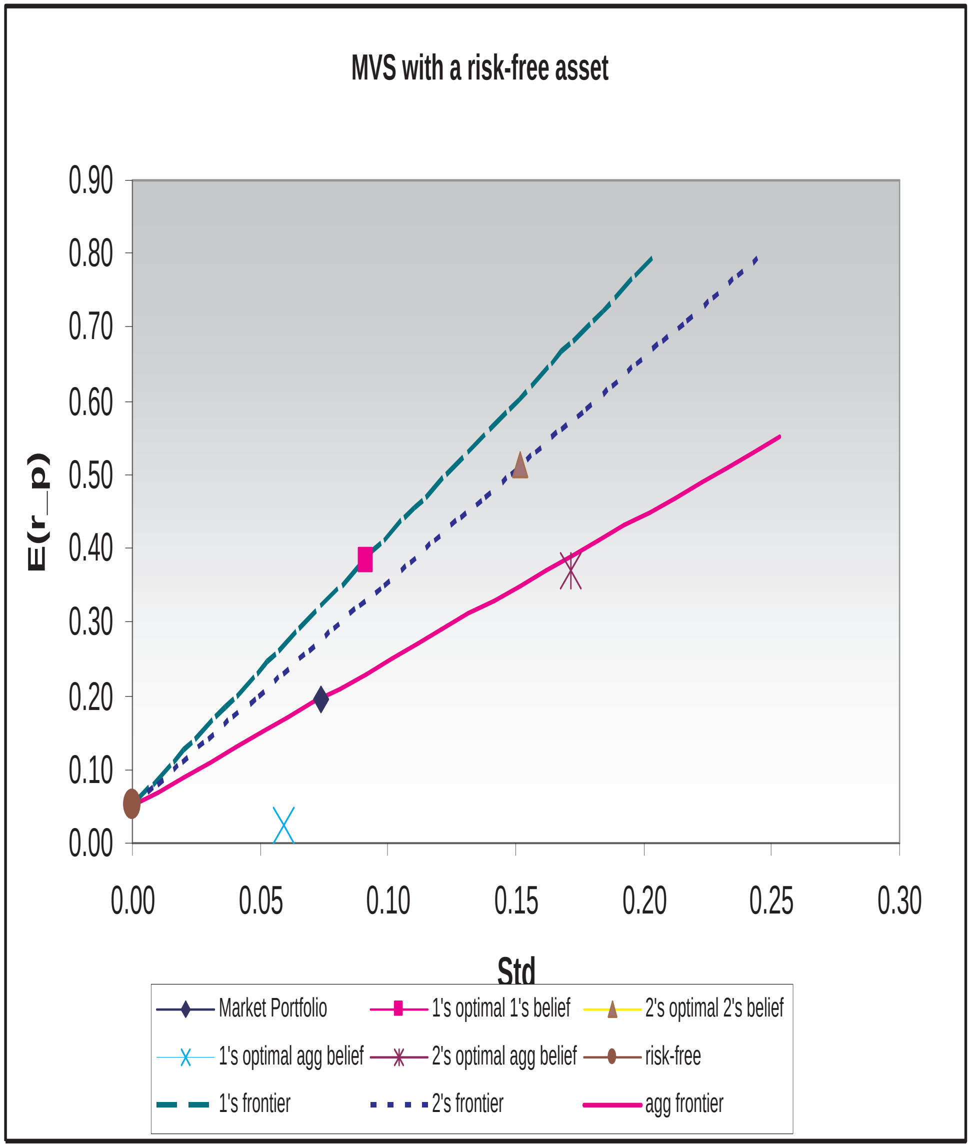
(b3) $(\theta_1, \theta_2) = (2, 4)$

6.3 Effect of heterogeneous expected payoffs and covariance matrices

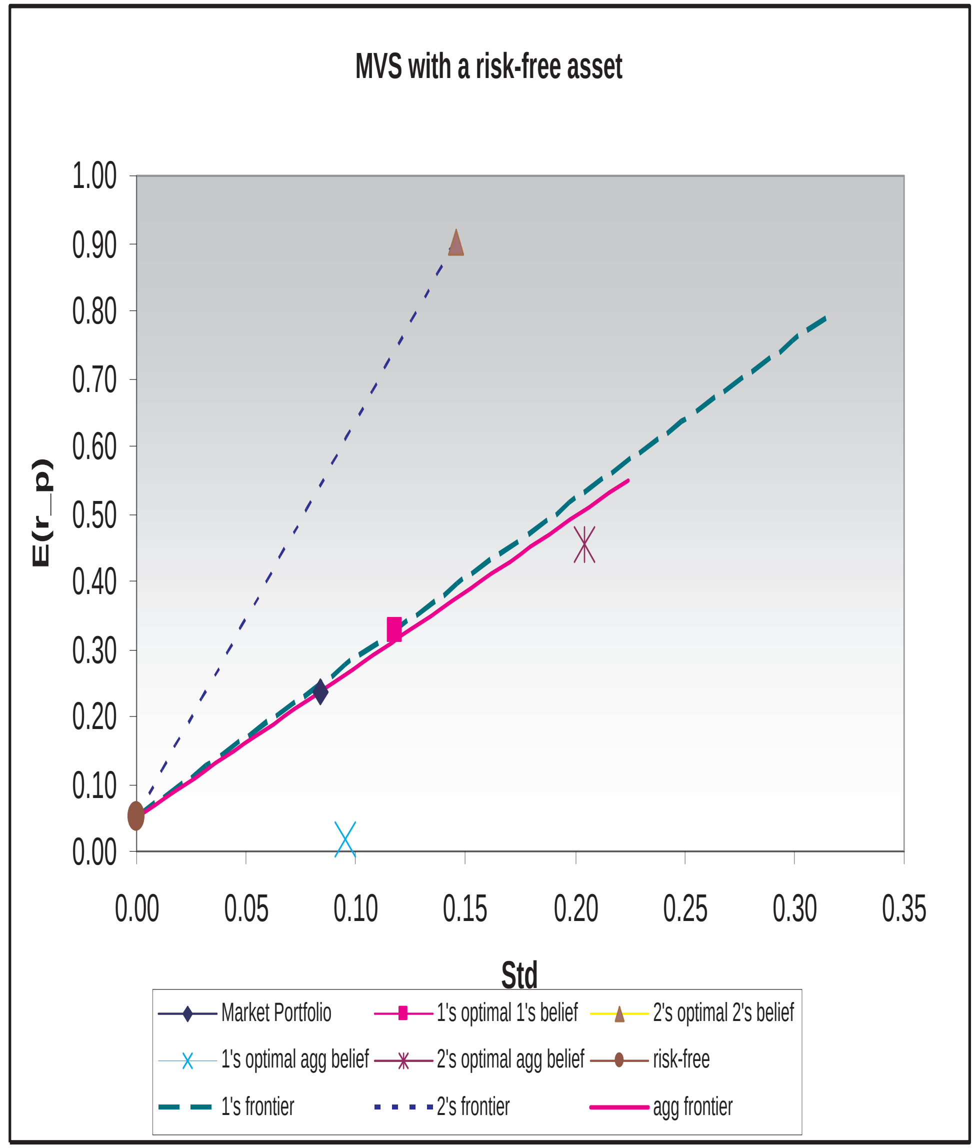
- **Example 3** Combining Examples 1 and 2 and consider two cases:
  (A) Investor 2 is optimistic in both the expected payoffs and covariance: $\mathbf{y_1} < \mathbf{y_2}$ and $\mathbf{\Omega_1} \geq \mathbf{\Omega_2}$.
  (B) Investor 2 is optimistic in the expected payoffs but pessimistic in covariance: $\mathbf{y_1} < \mathbf{y_2}$ and $\mathbf{\Omega_2} \geq \mathbf{\Omega_1}$.
- Implications for (A):
  - The standard one fund theorem does not held and the optimal portfolios become inefficient.
  - Surprisingly, the market's frontier is no longer in the middle, but below both individual frontier. The mean-variance efficiency of the optimal portfolios for both investors worsen.



$$(\theta_1, \theta_2) = (3, 3)$$

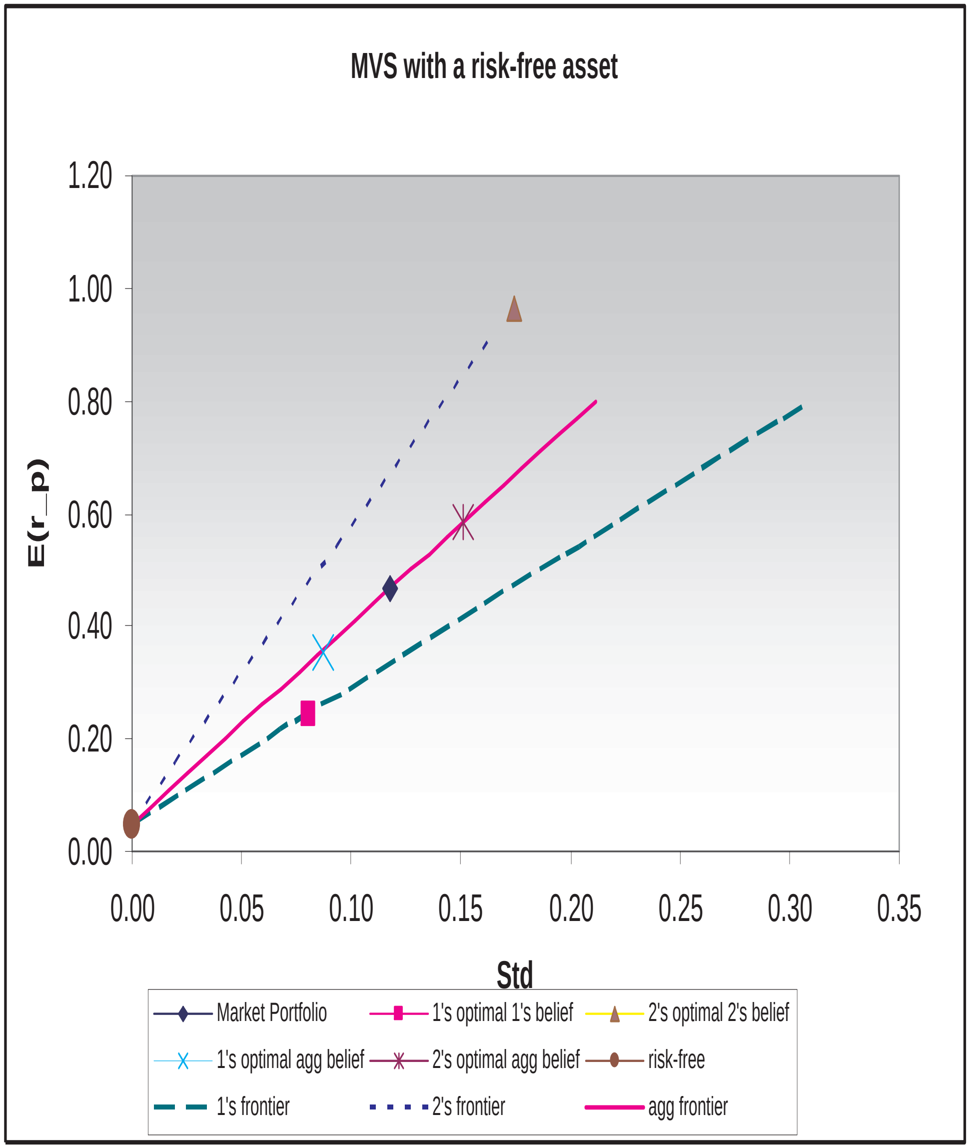


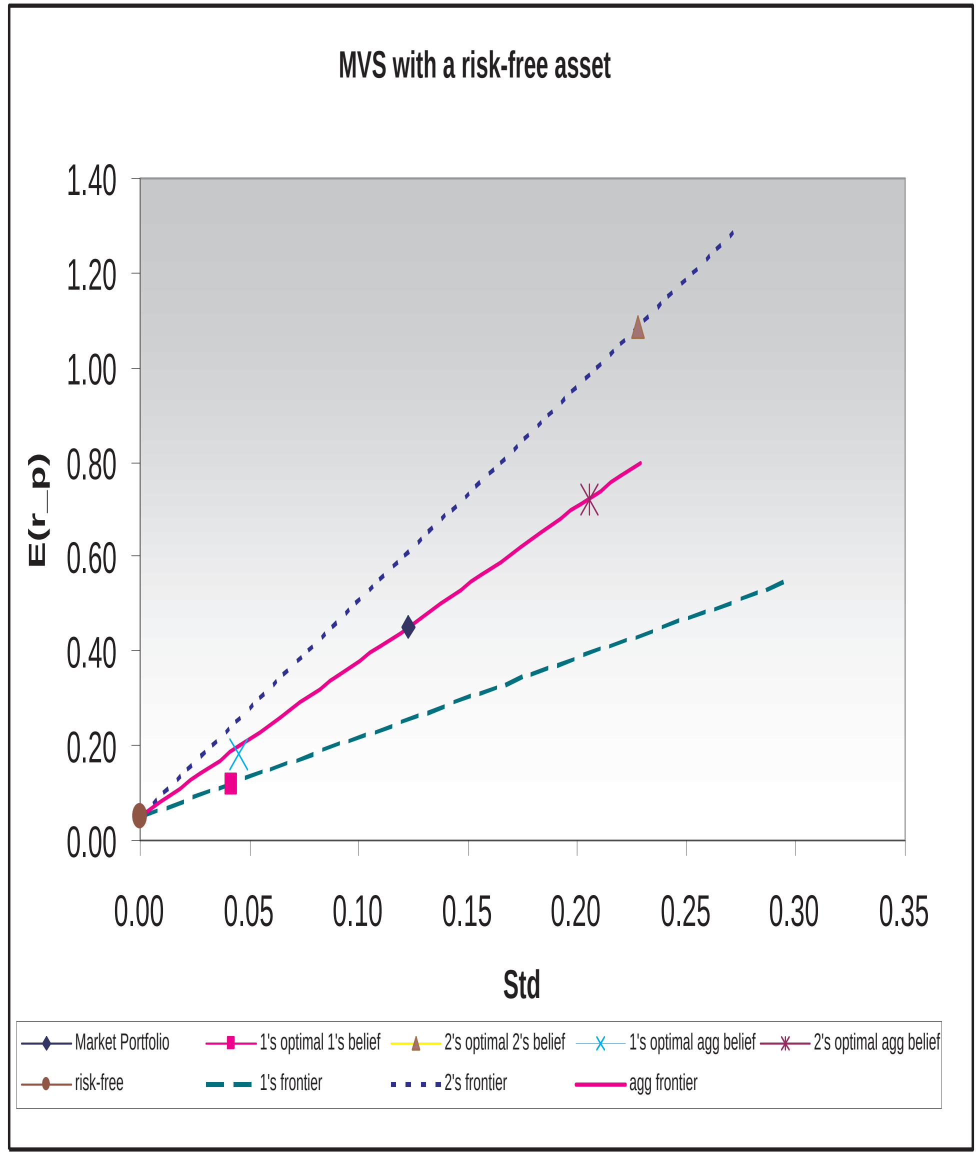
$$(\theta_1, \theta_2) = (4, 2)$$



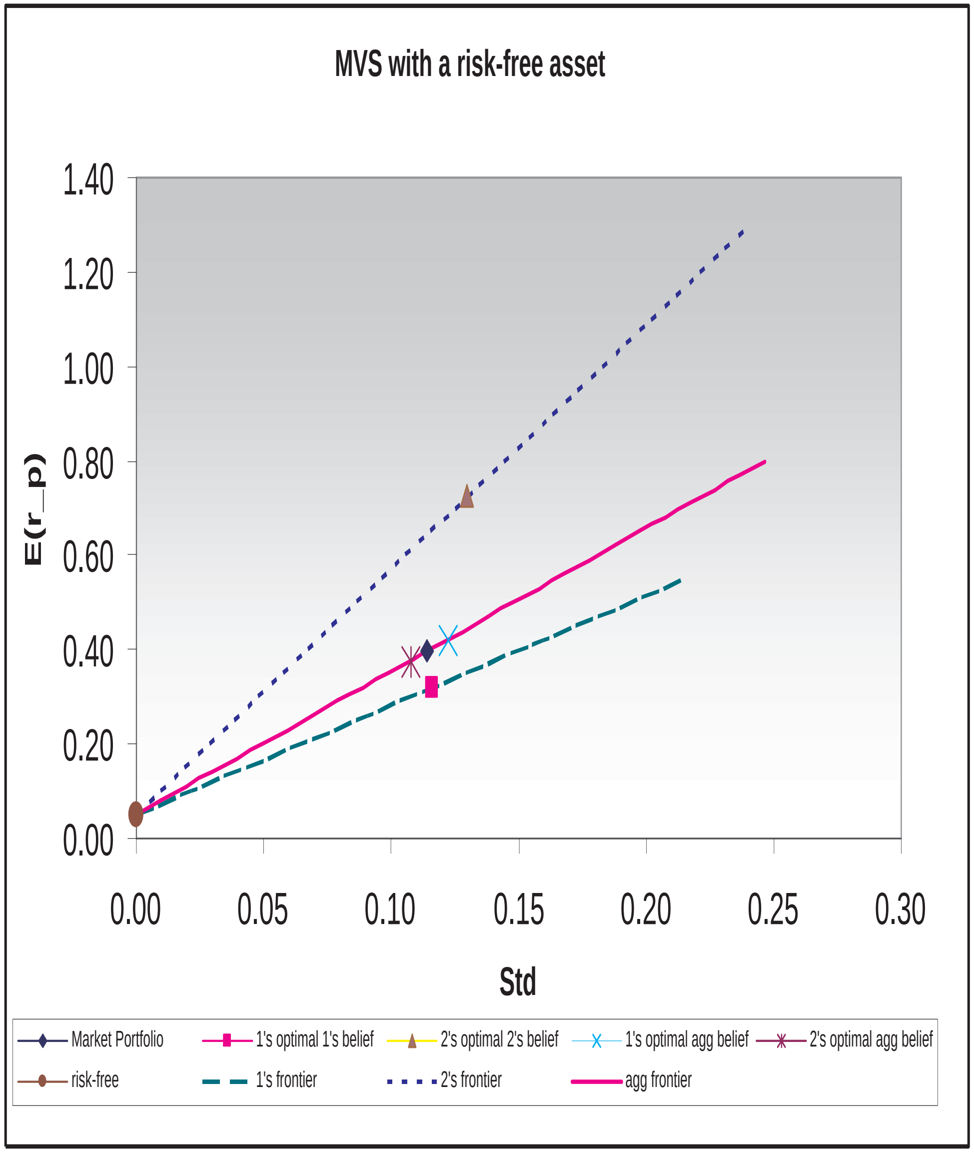
$$(\theta_1, \theta_2) = (2, 4)$$

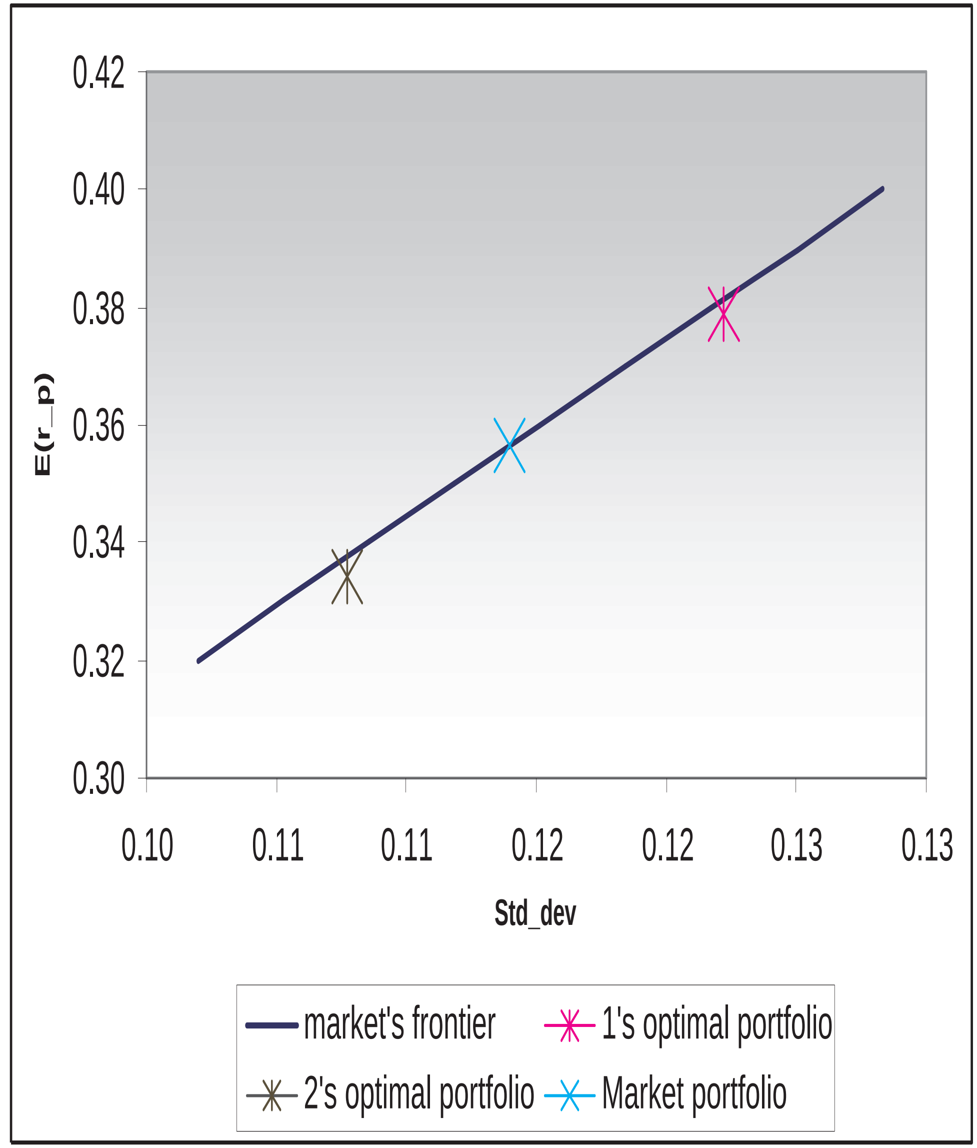
- Implications for (B):
  - The market frontier regained its position between the individual frontiers.
  - There is no mean-variance dominance for the optimal portfolios under the subjective and consensus beliefs.
- Overall:
  - the one fund theorem does not hold and the optimal portfolios of the investors becomes inefficient under the market belief.
  - The heterogeneity in covariance plays a very important role in determining the market frontier. This is because the aggregate return depends on not only the heterogeneous expected payoffs but also the covariance matrices.





$$(\theta_1, \theta_2) = (3, 3) \qquad (\theta_1, \theta_2) = (4, 2)$$





$$(\theta_1, \theta_2) = (2, 4)$$

$$(\theta_1, \theta_2) = (2, 4)$$

Summary:

- The standard one fund theorem under homogeneous belief does not hold under heterogeneous beliefs and the optimal portfolios become mean-variance inefficient under the market equilibrium belief.
- The heterogeneity in the covariance matrices plays the most important role in determining the relative positions of the individual frontiers and the market frontier, while the heterogeneity in expected payoffs plays the second important role.
- The risk aversion coefficients determine the relative positions of the individuals' optimal portfolios to the market portfolio.

7 Statistical Analysis on the Impact of the Heterogeneity

7.1 Impact of heterogeneous absolute risk aversion coefficients (ARA)

- **Observations**:
  - Except the case when $\sigma_{\theta_1} = \sigma_{\theta_2}$, the heterogeneity in ARAs generates non-normality in expected returns and betas.
  - The MPS in ARAs reduces expected return for all risky asset and Sharpe and Treynor ratios. The optimal portfolios of the two investors achieve the same Sharpe and Treynor ratios as the market portfolio under all scenarios.

- **Explanations**:
  - The market belief is the same as the investors' belief $\mathbf{y}_a = \mathbf{y}_i, \mathbf{\Omega}_a = \mathbf{\Omega}_i$ for $i = 1, 2$. Hence the one fund theorem under homogeneous belief is still hold. This explains the equal mean returns of the optimal portfolios of the investors and the market portfolio under all scenarios.
  - The aggregate market equilibrium price is a weighted average of each agent's equilibrium price under his/her belief as if he/she were the only agent in the market.

$$\mathbf{p}_{i,o} = \frac{1}{R_f}[\mathbb{E}_i(\tilde{\mathbf{x}}) - \theta_i \mathbf{\Omega}_i \bar{\mathbf{z}}_i],$$

$$\mathbf{p}_o = \frac{\Theta}{IR_f}\mathbf{\Omega}_a \Big[\sum_{i=1}^{I} \theta_i^{-1} \mathbf{\Omega}_i^{-1} \mathbb{E}_i(\tilde{\mathbf{x}}) - \bar{\mathbf{z}}_i\Big]. \quad (4)$$

  - The market equilibrium price is then dominated by the investor who is less risk averse, leading to higher market equilibrium prices and hence lower equilibrium returns.

- When σ_{θ_i} increases, the dominance of the less risk averse investors becomes significant, reducing the averages of the expected returns of the risky assets and hence portfolios.
- The convexity of $1/\theta_i$ leads to right skewed distributions for the market equilibrium prices and hence left skewed distributions for returns, leading to negative skewness for both returns and betas.

- **Conclusion**: The heterogeneity in risk aversion coefficients generates the non-normality in the market.

7.2 Impact of heterogeneous beliefs in expected payoffs

- **Observations**:
  - No significant change. The expected returns of the risky assets and optimal portfolios and the beta coefficients are normally distributed across all scenarios.
  - The Treynor ratios do not change at all. The Sharpe ratios are the same in all scenarios except for the optimal portfolios of investors, which decreases systematically as σ_{δ_i} increases. In addition, both the optimal portfolios have approximately the same Sharpe ratios, which are below the Sharpe ratio for the market portfolio.
- **Explanations**:
  - Note

$$\mathbf{y}_a = \frac{1}{2}(\mathbf{y}_1 + \mathbf{y}_2), \qquad \mathbf{p}_o = \frac{1}{R_f}[\mathbf{y}_a - \frac{\theta_o}{2}\Omega_o \mathbf{z}_m]$$

– On average, the MPS distribution in the expected asset payoffs does not change the market aggregate expected payoffs $\mathbf{y}_a$ and hence the equilibrium price.

– The insignificant standard deviations for the risky assets and betas lead to the same Treynor and Sharpe ratios under all scenarios.

– The under-performance of the optimal portfolios of the investors comparing to the market portfolio is due to their biased expected payoffs from the market.

- **Conclusion**: The heterogeneity in the expected payoffs has no significant effect on the market. However the unsystematic risk for the optimal portfolios of the investors increases as σ_{δ_i} increases and this is due to their bias towards the expected payoffs.

7.3 Impact of heterogeneous beliefs in covariance matrices of the asset payoffs

- In this case, $\mathbf{y} = \mathbf{y}_o$ and

$$\boldsymbol{\Omega}_a^{-1} = \frac{1}{2}(\boldsymbol{\Omega}_1^{-1} + \boldsymbol{\Omega}_2^{-1}].$$

- On average $\boldsymbol{\Omega}_a$ and hence the equilibrium prices are unchanged.
- Because of the convexity, the distributions of the expected returns of the risky assets and portfolios are skewed to the left.
- Overall, the heterogeneity in the covariance matrices has no significant impact on the market equilibrium returns, Sharpe and Treynor ratios, and normality of the distributions for the expected returns and beta coefficients. The optimal portfolios perform approximately equally to the market portfolio.

7.4 Impact of two or three sources of heterogeneity

- The heterogeneity in the risk aversion coefficients has significant impact on the market and it can generate non-normality of the expected returns and beats and both systematic and unsystematic risks, measured by the changes in the Sharpe and Treynor ratios.
- Overall it carry on the impact of the single source of heterogeneity. When the heterogeneity in ARAs is involved, the market is dominated by the heterogeneity in ARAs.
- In the case of heterogeneity in both expected payoffs and the covariance matrices, there is no significant impact on the market, although the impact of the heterogeneous beliefs in the expected payoffs dominates.

8 Summary

- The standard one fund theorem under homogeneous belief does not hold under heterogeneous beliefs and the optimal portfolios become mean-variance inefficient under the market equilibrium belief.
- The heterogeneity in the covariance matrices plays the most important role in determining the relative positions of the individual frontiers and the market frontier, followed by the heterogeneity in expected payoffs. The risk aversion coefficients determine the relative positions of the optimal portfolios to the market portfolio.
- MPS in heterogeneity in risk aversion coefficients, not the expected payoffs and covariance matrices, has significant impact on the market, and it can generate non-normality of the expected returns and beats and both systematic and unsystematic risks.

9 Other Setups

9.1 Heterogeneous Beliefs in Returns with a Risk-free Asset

9.1.1 Heterogeneous Beliefs and Consensus Belief

- **Market**: one frisk-free asset (r_f) and N risky assets ($\tilde{r}_j, j = 1, 2, \cdots, N$).
- **Heterogeneous Beliefs**
  - Some of the ideas go back to Lintner (1969).
  - Assume $\tilde{r}_j \sim MVN$
  - Heterogeneous beliefs $\mathcal{B}_i$ defined by $\mathcal{B}_i(\tilde{\mathbf{r}}) = (\mathbb{E}_i(\tilde{\mathbf{r}}), \Omega_i = Cov_i(\tilde{r}_k, \tilde{r}_l))$.
- **Optimal Portfolio**:
  - Investor i has a concave utility of wealth function $u_i(\cdot)$.

– Portfolio wealth: $\widetilde{W}_i = W_0^i(1 + r_f + w^T(\tilde{r} - r_f \mathbf{1}))$

– The global absolute risk aversion:

$$\theta_i := -E_i\left[u_i''(\widetilde{W}_i)\right] / E_i\left[u_i'(\widetilde{W}_i)\right]$$

– The optimal portfolio of investor i:

$$\mathrm{w}_i = \frac{\theta_i^{-1}}{W_0^i}\Omega_i^{-1}E_i\left[\widetilde{\mathrm{r}} - r_f\mathbf{1}\right].$$

- **Aggregation**:

  – Aggregate wealth

$$\sum_{i=1}^{I} W_0^i \mathrm{w}_i = \sum_{i=1}^{I} \theta_i^{-1}\Omega_i^{-1}E_i\left[\widetilde{\mathrm{r}} - r_f\mathbf{1}\right]$$

  – The vector of the *aggregate wealth proportions* in the risky assets

$$\mathbf{w}_a = \frac{1}{W_{m0}} \sum_{i=1}^{I} W_0^i \mathbf{w}_i = \frac{1}{W_{m0}} \sum_{i=1}^{I} \theta_i^{-1} \Omega_i^{-1} E_i \left[\tilde{\mathbf{r}} - r_f \mathbf{1} \right]$$

- **Consensus Belief**: $\mathcal{B}_a = \{\mathbb{E}_a(\tilde{\mathbf{r}}), \Omega_a\}$
  - Aggregate risk aversion: $\Theta := \left(\sum_{i=1}^{I} \theta_i^{-1}\right)^{-1}$.
  - An "aggregate" variance/covariance matrix Ω_a can be defined as

$$\Omega_a^{-1} = \Theta \sum_{i=1}^{I} \theta_i^{-1} \Omega_i^{-1}.$$

  - The "aggregate" expected returns on the risky assets $E_a(\tilde{\mathbf{r}})$:

$$E_a(\tilde{\mathbf{r}}) = \Theta \Omega_a \sum_{i=1}^{I} \theta_i^{-1} \Omega_i^{-1} E_i(\tilde{\mathbf{r}})$$

9.1.2 Equilibrium CAPM

- **Market Portfolio**:
  - We define the random return $\widetilde{r}_m$ on the market

$$\widetilde{W}_m := \sum_{i=1}^{I} \widetilde{W}_i = W_{m0}(1 + \widetilde{r}_m) \quad \Rightarrow \quad \boxed{\widetilde{r}_m = \frac{W_m}{W_{m0}} - 1}$$

  - In terms of aggregate wealth proportions $r_m := r_f + \mathbf{w}_a^\top (\mathbf{r} - r_f \mathbf{1})$
  - the aggregate 'consensus' variance belief: $\sigma^2_{a,m} := \mathbf{w}_a^\top \Omega_a \mathbf{w}_a$
  - Then the aggregate expected market return

$$\boxed{E_a(r_m) := r_f + \mathbf{w}_a^\top (E_a(\mathbf{r}) - r_f \mathbf{1})}$$

  - Aggregate variance of market portfolio becomes

$$\sigma^2_{a,m} = \frac{1}{\Theta W_{m0}} \quad \mathbf{w}_a^\top [E_a(\mathbf{r}) - r_f \mathbf{1}]$$

- **Return Relation**
  - The aggregate expected market risk premium is proportional to the aggregate relative risk aversion of the economy:

$$[E_a(r_m) - r_f] = \Theta W_{m0} \sigma_{a,m}^2$$

  - Aggregate excess return $\Omega_a \mathbf{w}_a = \frac{1}{\Theta W_{m0}}[E_a(\tilde{r}) - r_f.\mathbf{1}]$
  - The CAPM Equilibrium relation under the heterogeneous beliefs:

$$[E_a(\mathbf{r}) - r_f \mathbf{1}] = \frac{1}{\sigma_{a,m}^2} \Omega_a \mathbf{w}_a [E_a(r_m) - r_f].$$

- **Heterogeneous beta**:

$$\beta_{a,m} = \frac{\Omega_a \mathbf{w}_a}{\sigma_{a,m}^2} = \frac{[E_a(\mathbf{r}) - r_f \mathbf{1}]^\top \Omega_a^{-1} \mathbf{1}}{[E_a(\mathbf{r}) - r_f \mathbf{1}]^\top \Omega_a^{-1} [E_a(\mathbf{r}) - r_f \mathbf{1}]} [E_a(\mathbf{r}) - r_f \mathbf{1}]$$

9.1.3 Equilibrium Prices

- Assume that agents have CARA utility $\Rightarrow \theta_i$ = constant.
- In this case we obtain explicitly the optimal demands

$$\mathbf{w}_i = \frac{1}{W_0^i} \theta_i^{-1} \Omega_i^{-1} E_i \left[\mathbf{r} - r_f \mathbf{1} \right]$$

- The equilibrium price

$$\mathbf{p}_0 = \mathbf{Z}^{-1} \sum_{i=1}^{I} \theta_i^{-1} \Omega_i^{-1} E_i \left[\mathbf{r} - r_f \mathbf{1} \right]$$

where $\mathbf{z} := [z_1, z_2, ..., z_N]^T$ the supply vector and $\mathbf{Z} := diag[z_1, z_2, ..., z_N]$.

- The betas can also be expressed in terms of market clearing prices:-

$$\beta_{a,m} = \frac{\mathbf{p}_0^\top \mathbf{z}}{\mathbf{p}_0^\top \mathbf{Z} \Omega_a \mathbf{Z} \mathbf{p}_0} \Omega_a \mathbf{Z} \mathbf{p}_0$$

9.1.4 One fund theorem and mean-variance efficiency

Main Results:

- The standard one fund theorem under homogeneous belief does not hold under heterogeneous beliefs.
- The optimal portfolios under heterogeneous beliefs become mean-variance inefficient under the market aggregation. However, they are very close to the market frontier and hence quasi-one fund theorem holds under heterogeneous beliefs.
- Heterogeneity and bounded rationality lead to almost perfect rational market.

- Different aspects of the heterogeneity affect the market differently.
  - The heterogeneity in covariance plays the most important role in determining the relative positions of the individual and market frontiers.
  - The heterogeneity in the expected returns plays the second important role in determining the relative positions of the frontiers.
  - The risk aversion coefficients determine the closeness of the individuals' optimal portfolios to the market portfolio.
  - Depending on the combinations of different aspects of heterogeneity, the market can generate market risk premium ranging from below to above the risk premia of the individual optimal portfolios.
- A higher market risk premium is associated with a lower risk-free rate.

9.1.5 Statistic analysis of the aggregate market behaviour

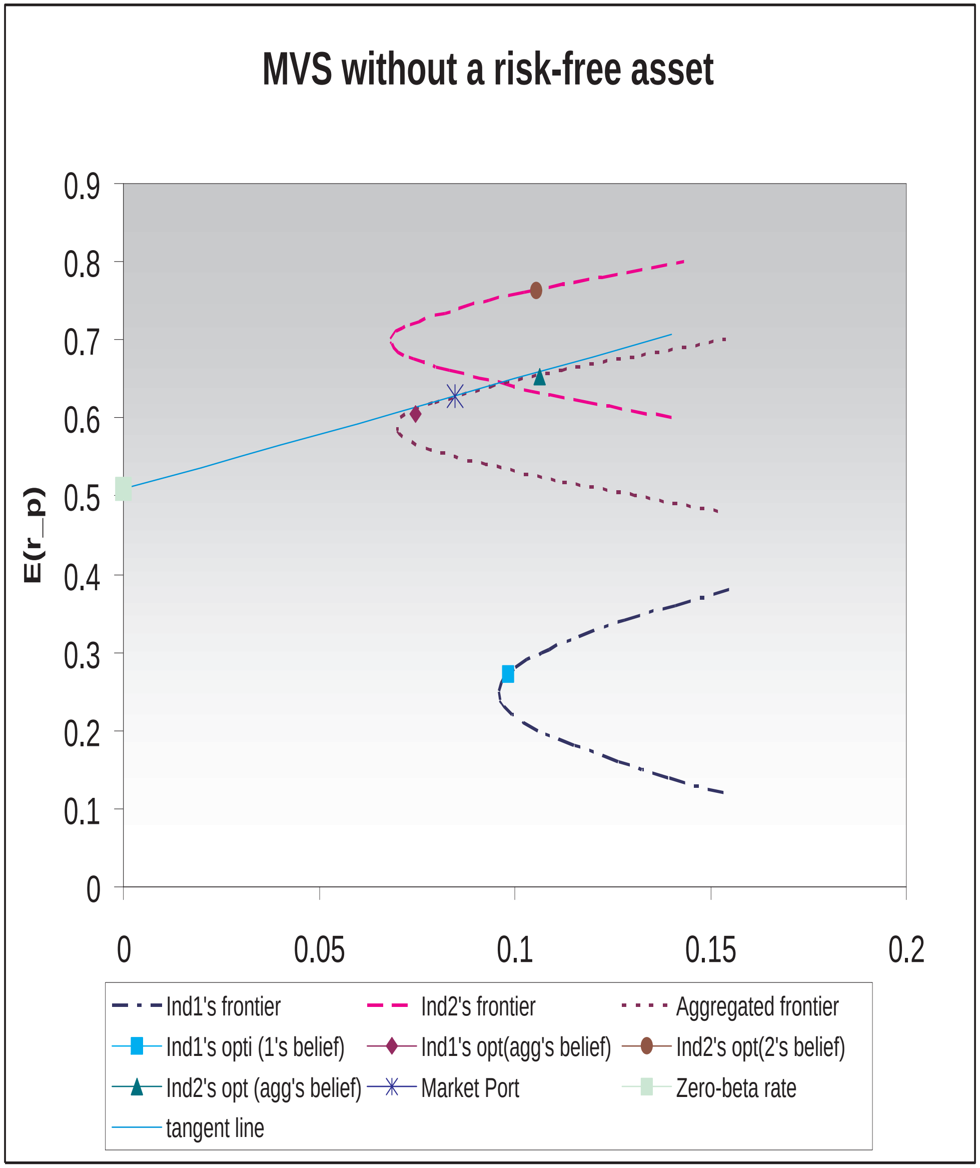
Main Results:

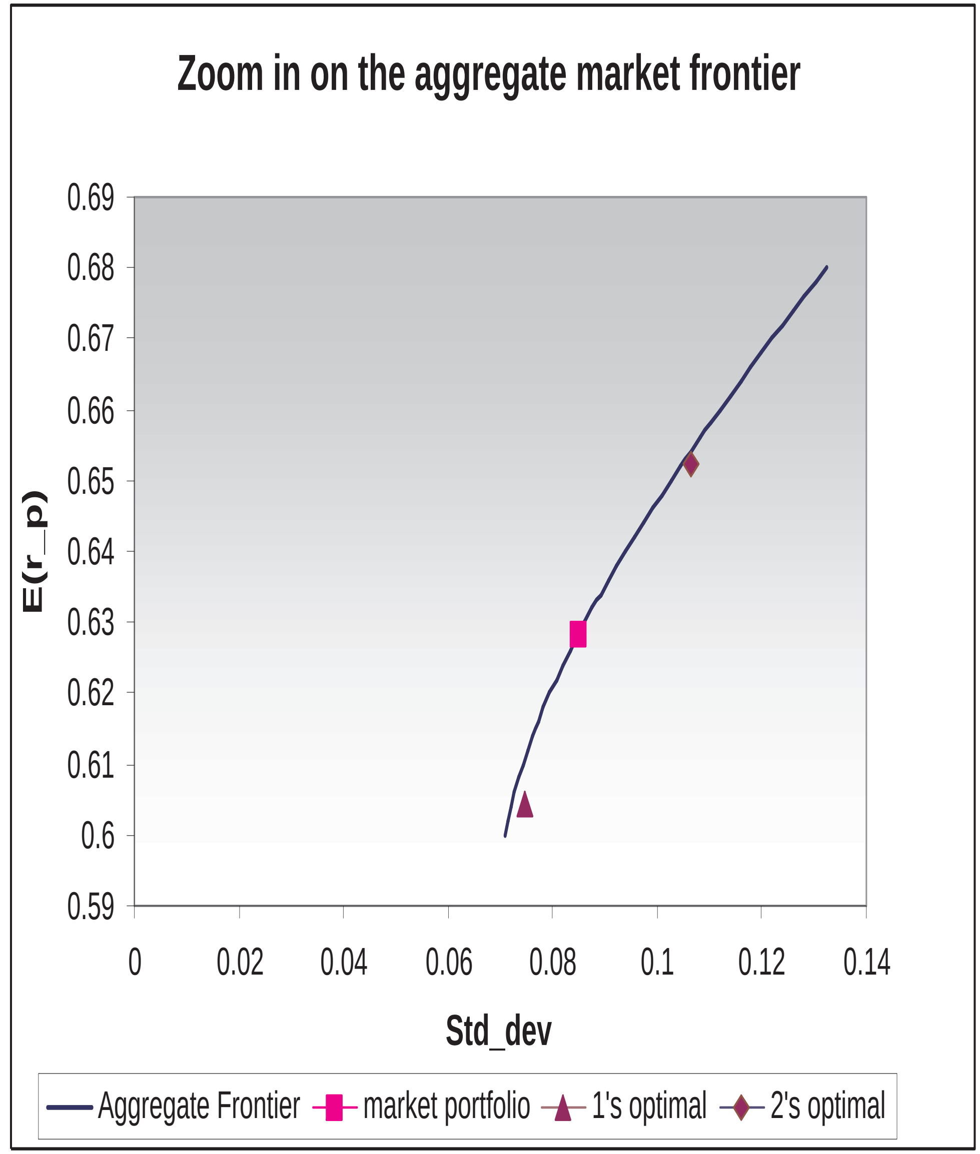
- Diversity leads to better outcomes:
  - The expected returns and standard deviations of assets and portfolios increase as the MPS increases.
  - The Sharpe and Treyon ratios increase systematically for the portfolios as the MPS increases.
- Market aggregation of heterogeneous leads to non-normality in return distributions.

9.2 Heterogeneous Beliefs in Payoffs without Risk-free Asset

Main Contributions:

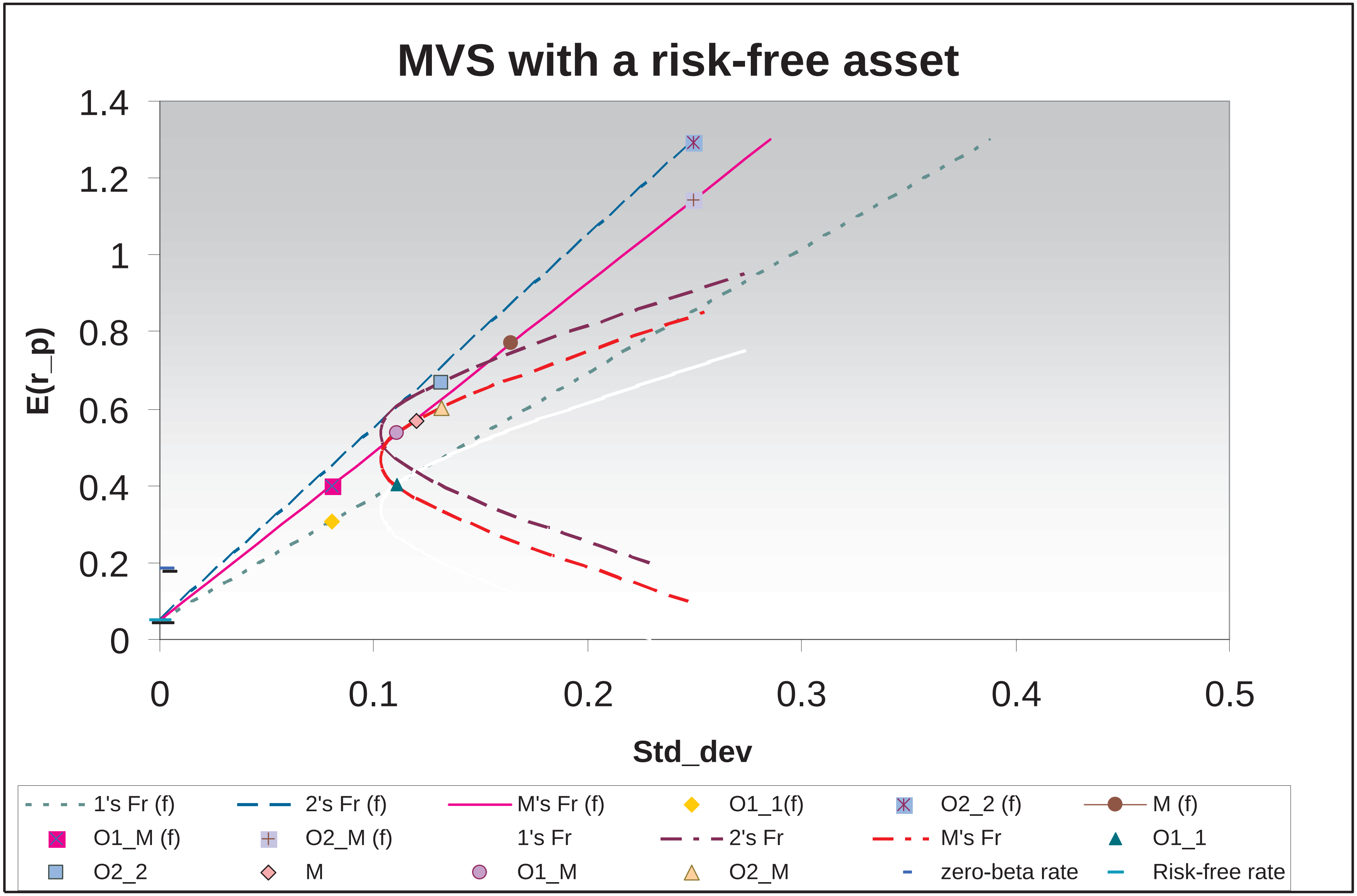
- Extends the standard Black's zero-beta CAPM with homogeneous beliefs to the case with heterogeneous beliefs.
- By introducing and constructing a consensus belief of the market, we show that Black's zero-beta CAPM holds under the consensus belief.
- The biased belief (from the market belief) of investor makes the optimal portfolio of the investor be mean-variance inefficient while the market portfolio is always on the efficient frontier.
- Theoretic fundation on the empirical finding that managed funds under-perform comparing to the market indices on average.


MVS without a risk-free asset
E(r_p)
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
0
0.05
0.1
0.15
0.2
Ind1's frontier
Ind2's frontier
Aggregated frontier
Ind1's opti (1's belief)
Ind1's opt(agg's belief)
Ind2's opt(2's belief)
Ind2's opt (agg's belief)
Market Port
Zero-beta rate
tangent line

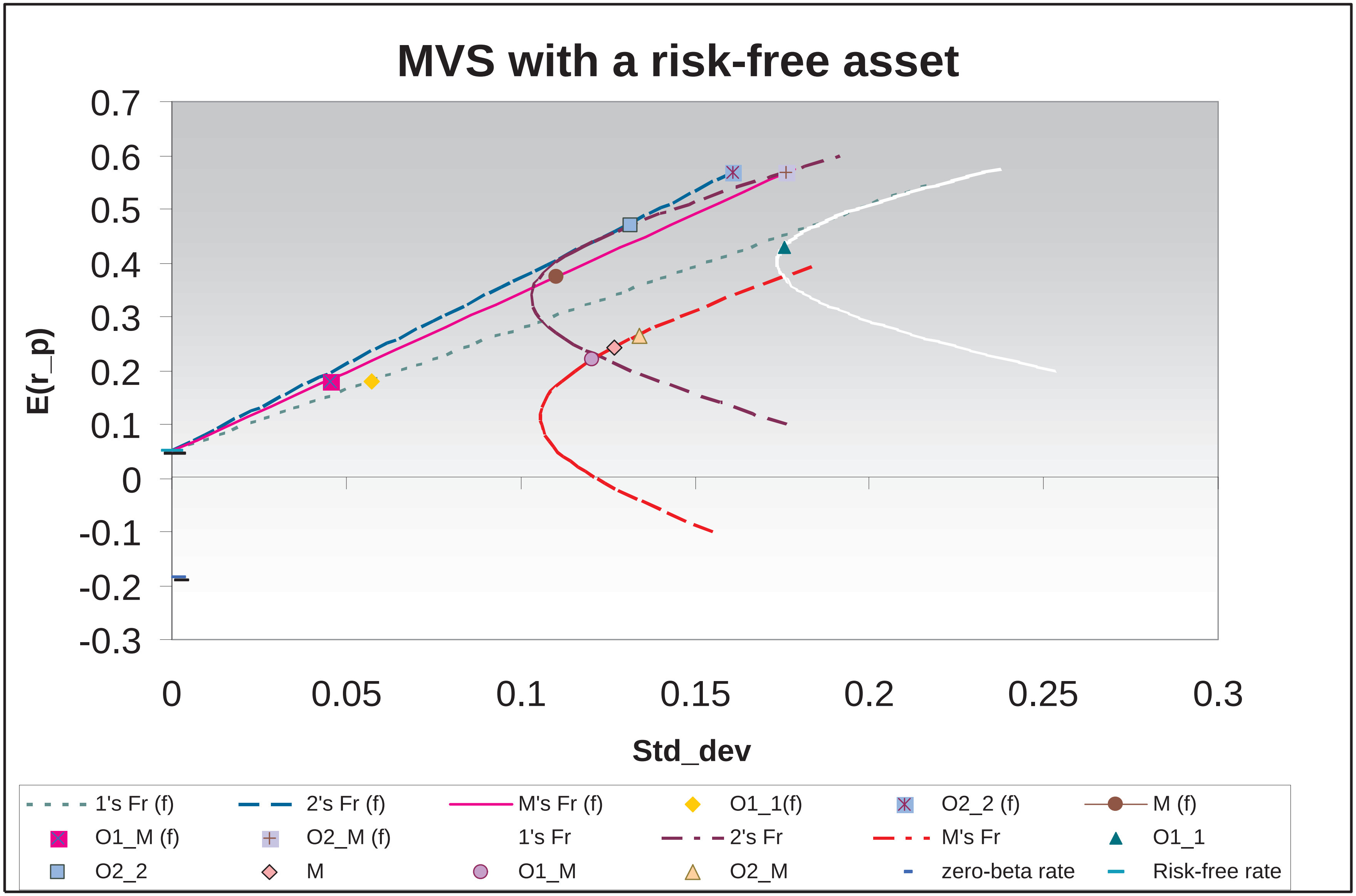

Zoom in on the aggregate market frontier
E(r_p)
0.69
0.68
0.67
0.66
0.65
0.64
0.63
0.62
0.61
0.6
0.59
0
0.02
0.04
0.06
0.08
0.1
0.12
0.14
Std_dev
Aggregate Frontier
market portfolio
1's optimal
2's optimal

9.3 Heterogeneous Beliefs in Returns without Risk-free Asset

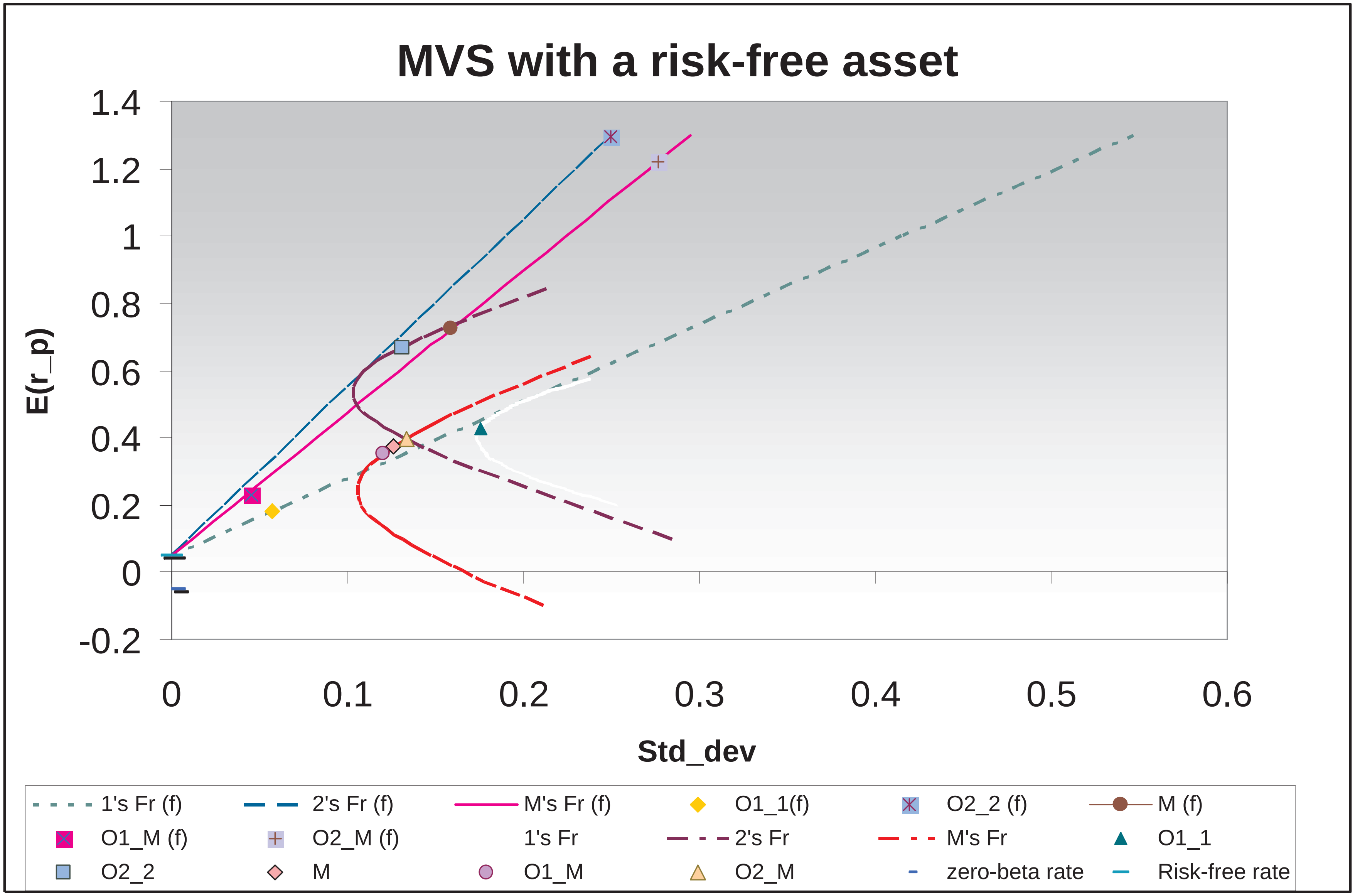
10 The Overall Geometric Relationship among Frontiers



$$\mu_1 < \mu_2, \Omega_1 = \Omega_2, \theta_1 = 4, \theta_2 = 2$$



$$\mu_1 = \mu_2, \Omega_1 > \Omega_2, \theta_1 = 4, \theta_2 = 2$$



$$\mu_1 < \mu_2, \Omega_1 > \Omega_2, \theta_1 = 4, \theta_2 = 2$$

11 Conclusions

- Provide a simple framework to aggregate the heterogeneous beliefs under the mean-variance framework.
- The (Zero-Beta) CAPM-like relations in both price and returns are extended to the case of heterogeneous beliefs.
- The market aggregation behaviors, including the risk aversion, aggregate variance/covariance matrix, the market expected payoff and the equilibrium price, are weighted average of heterogeneous individual behavior.
- The standard one fund theorem under homogeneous belief does not hold under heterogeneous beliefs and the optimal portfolios become mean-variance inefficient under the market equilibrium belief.
- Bounded rationality can lead to almost perfect rationality.
- Different heterogeneity plays different role.

- Different setup leads to different market impact under the MPS in heterogeneity:
  - Payoff Setup: Risk aversion coefficients, not the expected payoffs and covariance matrices, has significant impact on the market.
  - Return Setup: Diversity leads to better outcomes and market aggregation of heterogeneous leads to non-normality in return distributions
- The results can be used to explained some empirical results.

- **Future work**: Extension to a multi-period dynamic CAPM with heterogeneous beliefs and dynamic betas would help us to understand
  - **market behaviors**: including long swings of the market price away from the fundamental price, market boom and crash;
  - **stylized facts**: including herding, volatility clustering, long-range dependence, the risk premium puzzle and
  - the relation between cross-sectional volatility and expected returns;
  - equity risk premium and implication of diversification.