Fuzzy inductive reasoning, expectation formation and the behavior of security prices

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Abstract

This paper extends the Santa Fe Artificial Stock Market Model (SFASM) studied by LeBaron, Arthur and Palmer (1999, Journal of Economic Dynamics and Control 23, 1487–1516) in two important directions. First, some might question whether it is reasonable to assume that traders are capable of handling a large number of rules, each with numerous conditions, as is assumed in the SFASM. We demonstrate that similar results can be obtained even after severely limiting the reasoning process. We show this by allowing agents the ability to compress information into a few fuzzy notions which they can in turn process and analyze with fuzzy logic. Second, LeBaron et al. have reported that the kurtosis of their simulated stock returns is too small as compared to real data. We demonstrate that with a minor modification to how traders go about deciding which of their prediction rules to rely on when making demand decisions, the model can in fact produce return kurtosis that is comparable to that of actual returns data. © 2001 Elsevier Science B.V. All rights reserved.

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1. Introduction

Neoclassical financial market models are generally formulated so economic agents are able to logically deduce their price expectations. Real stock markets do not however typically conform to the severe restrictions required to guarantee such behavior. In fact, the actual market environment is usually much more ill-defined. The dilemma is that in an ill-defined environment the ability to exercise deductive reasoning breaks down, making it impossible for individuals to form precise and objective price expectations. This implies that participants in real financial markets must rely on some alternative form of reasoning to guide their decision making. We conjecture that individual reasoning in an ill-defined setting can be described as an inductive process in which individuals exhibit limitations in their abilities to process and condense information. The reasoning process we postulate is melded with a model of a market for risky securities. The model is dynamic, permitting us to examine the behavior of security prices and volume over simulated time. The objectives of this paper are to first, model an inductive reasoning process and second, to investigate its implications for aggregate market behavior and in particular for the behavior of security prices.¹

The core of the prediction process modeled is an inductive reasoning scheme in which individuals at best rely on fuzzy decision-making rules due to limits on their ability to process and condense information. The results reveal that our model has the potential to jointly explain numerous empirical regularities and puzzles observed in securities markets, some of which neoclassical models have struggled with but have not been able to fully address.

This paper extends the Santa Fe Artificial Stock Market Model (SFASM) studied by LeBaron et al. (1999) in two important directions (see also Arthur et al., 1997). First, some might question whether it is reasonable to assume that traders are capable of handling a large number of rules for the mapping of market states into expectations, each with numerous conditions, as is assumed in the SFASM. We demonstrate that similar results can be obtained even after severely limiting the reasoning process. We show this by allowing agents the ability to compress information into a few fuzzy notions which they can in turn process and analyze with fuzzy logic. Our approach is motivated by existing evidence that suggests human reasoning can be modeled as if the thought process is described by the application of fuzzy logic (Smithson, 1987; Smithson and Oden, 1999). Second, LeBaron et al. (1999) have reported that the kurtosis of their simulated stock returns is too small as compared to real data. We

¹ LeBaron (2000) presents a comprehensive review of the emerging literature that has come to be known as ‘agent-based computational economics/finance’. The present paper contributes to this literature. See also Brock and Hommes (1998).
demonstrate that with a minor modification to how traders go about deciding which of their prediction rules to rely on when making demand decisions, the model can in fact produce return kurtosis that is comparable to that of actual returns data. We motivate this latter modification by noting that negative group polarization attitudes can be triggered by specific events (Lamm and Myers, 1978; Shiller, 1989, Chapter 2). We introduce what is referred to here as a state of ‘doubt’ about which prediction rule is best. We allow this state to occur with only a very small probability. The prices and returns generated by the model under this structure exhibit characteristics, including measures of kurtosis, that are very similar to actual data.

The paper is organized as follows. In Section 2, we provide some intuition about why our model might better explain the regularities and transition dynamics we see in real financial markets. Section 3 describes the market environment and develops a model based on the genetic-fuzzy classifier system that characterizes our hypothesis about how investors formulate their price expectations. Section 3 also describes the experiments we have conducted. These experiments investigate the plausibility of our conjecture concerning the expectation formation process through an examination of its consequences for price and volume behavior. Section 4 presents results from computer simulations of a market inhabited by agents who behave in the manner described in Sections 2 and 3. Section 5 presents our conclusions.

2. Expectations formation and market created uncertainty

2.1. Market created uncertainty and volatility

The success of any model in characterizing the actual behavior of a system’s facets will ultimately hinge on incorporating those elements that are essential to insure that the model explains what it is intended that it explain. No one knows for certain what essential ingredients are necessary for explaining the obtuse behavior seen in security prices. Empirical evidence (see Shiller, 1989) seems to suggest however, that a certain type of market created uncertainty, intentionally sidestepped in neoclassical models (probably because of its analytical intractability, may be a potential explanation.

Specifically, by market created uncertainty we mean the uncertainty created as a result of the interactions (either directly and/or indirectly) among heterogeneous market participants who must learn to form their expectations in a market environment that is inherently ill-defined. The environment is ill-defined because for a market participant to logically deduce his expectations, he needs to know before hand what other market participants are expecting. But since this is exactly what every other market participant needs to do also, there is circularity in the reasoning process. The result is that no one will be able to
logically deduce his or her expectations. Consequently, each market participant will by necessity form his or her price expectations based on a subjective forecast of the expectations of all other market participants. But when this is the case, no one will be absolutely certain of what the market as a whole is expecting. As a result, the market can develop a life of its own and respond in ways that are not correlated with movements in fundamental values. In Arthur’s words,

... the sense he (referring to a market participant) makes of the Rorschach pattern of market information $I_t$ is influenced by the sense he believes others may make of the same pattern. If he believes that others believe the price will increase, he will revise his expectations to anticipate upward-moving prices (in practice helping validate such beliefs). If he believes others believe a reversion to lower values is likely, he will revise his expectations downward. All we need to have self reinforcing suspicions, hopes, and apprehensions rippling through the subjective formation of expectations (as they do in real markets) is to allow that $I_t$ contains hints — and imagined hints — of others’ intentions. (Arthur, 1995, p. 23)

Hence, the process of expectation formation under such circumstances can be tremulous. This view of the market is akin to that suggested by Keynes’s (1936, p.150). Keynes regarded security prices as ‘the outcome of the mass psychology of a large number of ignorant individuals,’ with professional speculators mostly trying to outguess the future moods of irrational traders, and thereby reinforcing asset price bubbles. In a similar vein, Dreman (1977, p. 99) maintains that individual investors, including professionals, do not form opinions on independently obtained information, ‘the thinking of the group’ heavily influences their forecasts of future events. Similar views have also been advanced by Black (1986), De Long et al. (1989, 1990), Shiller (1984, 1989), and Soros (1994).

What makes the problem even worse is that such behavioral uncertainties diminish the incentives for arbitrage which in turn impair the market’s natural tendency to return itself to a focus on fundamentals. In particular, these uncertainties create two types of risk for potential arbitrageurs which Shleifer and Summers (1990) have characterized as identification risk and noise trader risk (future resale price risk).

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3 We should also point out that Gennai and Leland (1990), and Jacklin et al. (1992) have demonstrated that uncertainty among market participants about the proportions of investors who follow various investment strategies is sufficient to produce market crashes, even if investors rationally update their beliefs over time.

4 See also Dreman (1982).
Identification risk arises because uncertainty in the market makes it difficult for potential arbitrageurs to distinguish between price movements driven by noise trader actions and price movements driven by pieces of private information which arbitrageurs have not yet received. Hence, it is difficult for potential arbitrageurs to exploit noise traders because they can never be completely sure that any observed price movement was driven by noise, which creates profit opportunities, or by news that the market knows but they have not yet learned.

In addition, there is also the risk that the price may systematically move away from its fundamental value because of noise trader activity. This latter risk is known as noise trader risk or future resale price risk. An investor who knows, even with certainty, that an asset is overvalued will still take only a limited short position because noise traders may push prices even further away from their fundamental values before the time to close the arbitrage position arrives.

Another type of risk, fundamental risk, although not due to the market uncertainty we have discussed, can also limit arbitrage. Fundamental risk is inherent in the market. It is the possibility that the fundamental value of the financial asset may change against the arbitrage position before the position is closed. Even if noise traders do not move prices away from fundamental values, changes in the fundamentals themselves might move the price against the investor.

Altogether, these problems make arbitrage risky and limit arbitrage activity. Arbitrage plays an ‘error-correction’ role in the market, bringing asset prices into alignment with their fundamental values. This role will be hampered when arbitrage is limited. As a result, asset prices may deviate from fundamental values and such deviations may persist, weakening whatever correlation there may have been between movements in asset prices and movements in their fundamental values.

The implication of this discussion is that when we impose the assumption of ‘mutual consistency in perceptions’ (Sargent, 1993) in rational expectations models, by fiat we eliminate the market created uncertainty outlined above. We allow the agents in our model the opportunity to form their expectations based upon individual subjective evaluations, thus restoring the potential for market created uncertainty to emerge. We turn next to a consideration of how humans process the immense amount of information that they are constantly bombarded with and how they reason in situations that are ill-defined. This conjecture forms the basis for the process by which investors form expectations in the model.

2.2. Fuzzy reasoning and induction

On a typical day, an immense amount of information flows into the stock market. Market participants typically have very little time to decipher these
Fuzzy logic is the brainchild of Zadeh (1962, 1965). We return to its application and the definition of fuzzy rules in the context of individual decision making in Section 3.3. For a more recent survey of applications of fuzzy set theory in psychology, see Smithson and Oden (1999).

Data. How, therefore, should we characterize the process by which market participants assimilate such information flows? Some psychologists have argued that our ability to process an immense amount of information efficiently is the outcome of applying fuzzy logic as part of our thought process. This would entail the individual compressing information into a few fuzzy notions that in turn are more efficiently handled, and then reasoning as if by the application of fuzzy logic. But for us to have confidence in using fuzzy set theory as a model of the way humans think there are two questions that need to be addressed. The observations of Smithson (1987) lead us, which we paraphrase next. First, is there evidence that supports the hypothesis that at least some categories of human thought are fuzzy? Second, are the mathematical operations of fuzzy sets as prescribed by fuzzy set theory a realistic description of how humans manipulate fuzzy concepts? Smithson (1987) finds that the evidence on reasoning and the human thought process provides affirmative answers to both of these questions. Modeling human reasoning as if the thought process is described by the application of fuzzy logic takes us halfway toward our goal. We turn now to a discussion of the other important element in the process, induction.

We have earlier alluded to the fact that deductive reasoning will break down in an environment that is ill-defined. But if deductive reasoning will not work, how then can individuals form their expectations? Arthur et al. (1997) argue that individuals will form their expectations by induction (see also Arthur, 1991, 1992; Blume and Easley, 1995; Rescher, 1980). Rescher has defined induction as follows:

Induction is an ampliative method of reasoning — it affords means for going beyond the evidence in hand in endeavor to answer our questions about how things stand in the world. Induction affords the methodology we use in the search for optimal answers.

Induction as a cognitive method proceeds by way of the systematization of question-resolving conjecture with experience, by fitting conjectural extensions sufficiently tightly into the overall setting of our other (generally tentative) commitments. Though induction always involves a leap beyond the information in hand, it only endorses these leaps when the fit is sufficiently close. (Rescher, 1980, p. 87)
Simply put, induction is a means for finding the best available answers to questions that transcend the information at hand. In real life, we often have to draw conclusions based upon incomplete information. In these instances, logical deduction fails because the information we have in hand leaves gaps in our reasoning. In order to complete our reasoning we fill those gaps in the least risky, minimally problematic way, as determined by plausible best-fit considerations. Consequently, the conclusions we draw using induction are suggested by the data at hand rather than logically deduced from them. Nonetheless, induction should not be taken as mere guesswork; it is responsible estimation in the sense that we are willing to commit ourselves to the tenability of the answer that we put forth. In other words, we must find the answers to be both sensible and defensible.

Inductive reasoning follows a two-step process: possibility elaboration and possibility reduction. The first step involves creating a spectrum of plausible alternatives based on our experience and the information available. In the second step, these alternatives are tested to see how well they answer ‘the question’ at hand or how well they connect the existing incomplete premises to explain the data observed. The alternative offering the best fit connection is then accepted as a viable explanation. Subsequently when new information becomes available or when the underlying premises change, the fit of the current connection may degrade. When this happens a new best fit alternative will take over.

How can induction be implemented in a securities market model? Arthur et al. (1997) visualize induction in securities markets taking place as follows. Under this scheme of rationalizing, each individual in the market continually creates a multitude of ‘market hypotheses’ (this corresponds to the possibility elaboration step discussed above). These hypotheses, which represent the individuals’ subjective expectational models of what moves the market price and dividend, are then simultaneously tested for their predictive ability. The hypotheses identified as reliable will be retained and acted upon in buying and selling decisions. Unreliable hypotheses will be dropped (this corresponds to the possibility reduction step) and ultimately replaced with new ones. This process is carried out repeatedly as individuals learn and adapt in a constantly evolving market.

The expectations formation process we have just described can be adequately modeled by letting each individual form his expectations using his own personal genetic-fuzzy classifier system. Each genetic-fuzzy classifier system contains a set of conditional forecast rules that guide decision making. We can think of these rules as the subjective ‘market hypotheses’ held by each individual. Inside the classifier system is a genetic algorithm that is responsible for generating new rules, testing all existing rules, and weeding out bad rules. The possibility-elaboration step is then captured by the constant formulation of new conditional forecast rules in the system; the subsequent testing of these conditional forecast rules and the eventual removal of bad ones represent the possibility-reduction step.
3. Models and experiments

3.1. The market environment

The basic framework we study is similar to the Santa Fe Artificial Stock Market (see Arthur et al. (1997) or LeBaron et al. (1999)). The market considered here is a traditional neoclassical two-asset market. However, we deviate from the traditional model by allowing agents to form their expectations inductively using a genetic-fuzzy classifier system, which we outline in the next section.

There are two tradable assets in the market, a stock that pays an uncertain dividend and a risk-free bond. We assume that the risk-free bond is in infinite supply and that it pays a constant interest rate $r$. There are $N$ units of the risky stock, and each pays a dividend of $d_t$. The dividend is driven by an exogenous stochastic process $\{d_t\}$. The agents who form the core of the model do not know the structure of the dividend process. We follow LeBaron et al. (1999) and assume that dividends follow an AR(1) mean reverting dividend process of the form,

$$d_t = \bar{d} + \rho (d_{t-1} - \bar{d}) + \varepsilon_t,$$

(1)

where $\varepsilon_t$ is Gaussian, i.i.d., has zero mean, and variance $\sigma^2_\varepsilon$. The subscript $t$, represents time. We assume that time is discrete and that investors solve a sequence of single period problems over an infinite horizon. Each agent in the model is initially endowed with a fixed number of shares, which for convenience, we normalize to one share per agent.

There are $N$ heterogeneous agents in the market, each characterized by a constant absolute risk aversion (CARA) utility function of the form $U(W) = -\exp(-\lambda W)$ where $W$ is wealth and $\lambda$ represents degree of relative risk aversion. Agents, however, are heterogeneous in terms of their individual expectations and will only hold identical beliefs by accident. Thus, they will quite likely have different expectations. At each date, upon observing the information available to them, agents decide what their desired holdings of each of the two assets should be by maximizing subjective expected utility of next period wealth.

Assuming that agent $i$’s predictions at time $t$ of the next period’s price and dividend are normally distributed with mean, $\hat{E}_{i,t}[p_{t+1} + d_{t+1}]$ and variance, $\hat{\sigma}^2_{i,t,p+d}$, then agent $i$’s demand, $x_{i,t}$, for shares of the risky asset is given by

$$x_{i,t} = \frac{\hat{E}_{i,t}[p_{t+1} + d_{t+1}] - p_t(1+r)}{\lambda \hat{\sigma}^2_{i,t,p+d}},$$

(2)
where $p_t$ is the price of the risky asset at time $t$, and $\lambda$ is the degree of relative risk aversion.\footnote{The optimal demand function is derived from the first-order condition of expected utility maximization of agents with CARA utility under the condition that the forecasts follow a Gaussian distribution (see Grossman (1976), for details). However, when the distribution of stock prices is non-Gaussian (as we will see in our simulations) the above connection to the maximization of a CARA utility function no longer exists, so in these cases we simply take this demand function as given.} Agents in the model know that Eq. (2) will hold in a heterogeneous rational expectations equilibrium. However, the fact that they must use induction and that they use fuzzy rules when forming expectations, means that they never know if the market is actually in equilibrium. We assume that because of the ill-defined nature of the economic setting, agents always select to use (2) when setting their demands, knowing that sometimes the market will be in equilibrium and that sometimes it will not. Since total demand must equal the total number of shares issued, for the market to clear, we must have
\[ \sum_{i=1}^{N} x_{i,t} = N. \] (3)

### 3.2. The sequence of events

We now turn our attention to the timing of the various events in the model. The current dividend, $d_t$, is announced and becomes public information at the start of time period $t$. Agents then form their expectations about the next period’s price and dividend $E_{i,t}[P_{t+1} + d_{t+1}]$ based on all current information regarding the state of the market (which includes the historical dividend sequence $\{\ldots, d_{t-2}, d_{t-1}, d_t\}$ and price sequence $\{\ldots, p_{t-2}, p_{t-1}\}$). Once their price expectations are established, agents use Eq. (2) to calculate their desired holdings of the two assets. The market then clears at a price $p_t$.\footnote{Demand schedules are in turn conveyed to a Walrasian auctioneer who then declares a price $p_t$ that will clear the market.} The sequence of events is then repeated and prices at $t+1$ are determined, etc. The agents monitor the forecasting effectiveness of the genetic-fuzzy classifiers they have relied upon to generate their expectations. At each time-step the agents learn about the effectiveness of these classifiers. The classifiers that have proved to be unreliable are weeded out to make room for classifiers with new and perhaps better rules.

### 3.3. Modeling the formation of expectations

#### 3.3.1. Components of the reasoning model

We have already argued that individual reasoning in ill-defined environments can be described as a process of applying fuzzy logic and induction to the
formation of expectations. We operationalize this hypothesis by characterizing individual reasoning as if it was the product of the application of a genetic-fuzzy classifier system. Our genetic-fuzzy classifier is based in part on the design of the classifier system originally developed by Holland (see Goldberg (1989), Holland and Reitman (1978), or Holland et al. (1986)). At the heart of Holland’s classifier system are three essential components: a set of conditional action rules, a credit allocation system for assessing the predictive capability of any rule and a genetic algorithm (GA) by which rules evolve. The behavior of a classifier system is ultimately determined by its rules. Each rule contains a set of conditions and an action or a combination of actions. Whenever the prevailing state of the environment matches all the conditions in a rule, the system adopts the actions prescribed by the rule. The function of the credit allocation system is to systematically keep track of the relative effectiveness of each rule in the classifier. This information is in turn used to guide a GA in the invention of new rules and the elimination of ineffective rules. Together these components make it possible for the system, in our case the investor, to learn about the environment and adapt to innovations in the environment.\(^9\)

The genetic-fuzzy classifier we have employed to model expectation formation is a modification of Holland’s classifier system. In the spirit of our earlier discussion of fuzzy decision making, we replace the conventional rules in Holland’s classifier with fuzzy rules to create our genetic-fuzzy classifier. These fuzzy rules also involve a condition-action format but they differ from conventional rules in that fuzzy terms rather than precise terms now describe the conditions and actions.

### 3.3.2. The specification of forecasting rules

The object of interest in defining demand is the expectation of price plus dividend. Expectation formation is assumed to occur in the following manner. First, the ‘action’ part of each rule employed by an agent is identified with a set of forecast parameters.\(^{10}\) That is, if certain conditions prevail, the agent assigns certain values to the parameters of their expectation model. Second, these forecast parameters are substituted into a linear forecasting equation to generate the expectations desired. The forecast equation hypothesis is:

$$E_t(p_{t+1} + d_{t+1}) = a(p_t + d_t) + b,$$

\(^9\) Learning in the model is in the same spirit as the learning frameworks employed by Bray (1982), Marcet and Sargent (1989a, b) and Sargent (1991), among others, in which the prediction model used by an agent is misspecified except when the system is in equilibrium.

\(^{10}\) Arthur et al. (1997) employ a similar approach.
where $a$ and $b$ are the forecast parameters to be obtained from the activated rule, and the variables $p_t$ and $d_t$ are the price and dividend at time, $t$. The linear forecasting model shown in Eq. (4) is optimal when (a) agents believe that prices are a linear function of dividends and (b) a homogeneous rational expectations equilibrium obtains (see the Appendix for a proof of this assertion). While we place no such restrictions on the system studied here, a linear forecasting model serves as an approximation for the structure likely to evolve over time.

The format of a rule when fuzzy rules prevail, would therefore be ‘If specific conditions are satisfied then the values of the forecast equation parameters are defined in a relative sense’. One example is ‘If {price/fundamental value} is low, then $a$ is low and $b$ is high’.

A vector of relative information bits characterizes each rule. We use five information bits to specify the conditions in a rule. These bits represent five market descriptors. These descriptors include one factor that conveys information about the security’s fundamental value and four technical factors that convey information about price behavior (such as trends). We use another two bits to represent the forecast parameters $a$ and $b$. Altogether, we use a string of seven bits to represent each conditional forecast rule.

### 3.3.3. Coding market conditions

The five market descriptors we use for the conditional part of a rule are: $[p*r/d, p/MA(5), p/MA(10), p/MA(100), and p/MA(500)]$. The variables $r$, $p$ and $d$ are the interest rate, price, and dividend, respectively. The variable $MA(n)$ in the denominator denotes an $n$-period moving average of prices. We organize the positions of the five bits so that they refer to the market descriptors in the same order as above. Thus, the first information bit reflects the current price in relation to the current dividend and it indicates whether the stock is above or below the fundamental value at the current price. Clearly this is a ‘fundamental’ bit. The remaining four bits, bits 2–5, are ‘technical’ bits which indicate whether the price history exhibits a trend or similar characteristic. The utility of any information bit as a predictive factor determines its survival in a prediction rule.

Market descriptors are transformed into fuzzy information sets by first defining a range of possibilities for each information bit, and second, by

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11 We use (4) as the forecast equation because the homogenous linear rational expectation equilibrium solution is of this form and it allows us to investigate whether the agents are capable of learning this equilibrium solution.

12 In practice, we have two additional bits to denote, the weight of each rule in a Rule Base, and the logical connective used in the conditions of the rules. In our experiment, we have kept the weights the same and we have used the logical ‘AND’ operator in all the rules.

13 In other words, the five bits for the conditional part of a rule would be arranged according to: $[p*r/d, p/MA(5), p/MA(10), p/MA(100), p/MA(500)]$. 
specifying the number and types of fuzzy sets to use for each of these market variables.\textsuperscript{14} We set the range for each of these variables to \([0, 1]\).\textsuperscript{15} Within these limits we will assume that each market descriptor can be characterized by what we will call a membership function.

A membership function is a description of the possible states of a descriptor. We assume that each descriptor has the possibility of falling into four alternative states: ‘low’, ‘moderately low’, ‘moderately high’ and ‘high’. We let the possible states of each market descriptor be represented by a set of four membership functions that are associated with a specific shape. The state ‘low’ is associated with a trapezoidal shaped membership function that has its southwest corner at the origin. The state ‘moderately low’ is associated with a triangular shaped membership function that has a base stretching from a point slightly greater than 0 to a point slightly greater than 0.6. The state ‘moderately high’ is also associated with a triangular membership function, but its base runs from a point slightly less than 0.4 to a point slightly less than 1. Finally the state ‘high’ is associated with a trapezoidal membership function that has its southeast corner at the point 1 on the x-axis.\textsuperscript{16} The shapes and locations of these fuzzy sets along the specified range \([0, 1]\) (universe of discourse) are as illustrated in Fig. 1. We represent fuzzy information with the codes ‘1’, ‘2’, ‘3’ and ‘4’ for ‘low’, ‘moderately low’, ‘moderately high’ and ‘high’, respectively. A ‘0’ is used to record the absence of a fuzzy set.

An example will help at this point. If the conditional part of the rule is coded as \([0\ 1\ 3\ 0\ 2]\), this would mean that \(p*r/d\) and \(p/MA(100)\) are not present in the conditional part of the rule, that is, the rule assigns no influence to these descriptors. In contrast \(p/MA(5)\), \(p/MA(10)\) and \(p/MA(500)\) all have an influence and the coding indicates that \(p/MA(5)\) is ‘low’, \(p/MA(10)\) is ‘moderately high’ and \(p/MA(500)\) is ‘moderately low’. In other words, this corresponds to a state in which the market price is less than \(MA(5)\) but somewhat greater than \(MA(10)\) and is slightly less than \(MA(500)\). As long as the prevailing market situation matches the conditions for \(p/MA(5)\), \(p/MA(10)\), and \(p/MA(500)\), the conditional part of the rule will be fulfilled and this rule will be activated regardless of what the values for \(p*r/d\) and \(p/MA(100)\) might be.\textsuperscript{17}

\textsuperscript{14}The literature on fuzzy sets identifies the ‘range’ as we use it here with the terminology ‘universe of discourse.’

\textsuperscript{15}When we set the universe of discourse to the interval \([0, 1]\), we have implicitly multiplied each of the market descriptors by 0.5. So if a market descriptor is equal to 0.5, it means that market price is exactly equal to the benchmark (i.e. \(d/r, MA(5), \ldots\)) that is referred to in the market descriptor.

\textsuperscript{16}The terminology ‘membership function’ and ‘fuzzy set’ are used interchangeably in the literature to represent the same thing.

\textsuperscript{17}Therefore, ‘0’ has the same effect as the wildcard symbol ‘#’ in the SFASM model.
A critical point to remember is that while these market descriptors are intrinsically fuzzy, the conditions described by them are likely to match many different market situations. What really matters however, is the degree to which each of these conditions is fulfilled and not so much whether each condition is indeed matched or not matched by the prevailing market situation.

3.3.4. Identifying expectation model parameters

Now we turn to the modeling of the forecast part of the rule. We allow the possible states of each forecast parameter to be represented by four fuzzy sets. These fuzzy sets are labeled respectively (with the shapes indicated), ‘low’ (reverse S-curve), ‘moderately low’ (Gaussian-bell curve), ‘moderately high’ (Gaussian-bell curve), and ‘high’ (S-curve). The universe of discourse for the parameters, $a$ and $b$ are set to $[0.7, 1.2]$ and $[-10, 19]$, respectively.\(^{18}\) The shapes and locations of the fuzzy membership functions for $a$ and $b$ are as illustrated in Figs. 2 and 3. When we represent these fuzzy sets as bits, we code them ‘1’, ‘2’, ‘3’, and ‘4’ for ‘low’, ‘moderately low’, ‘moderately high’, and ‘high’ respectively.

An example of the forecast part of the rule is the string $[2\ 4]$, which would indicate that the forecast parameter $a$ is ‘moderately low’ and $b$ is ‘high’. Following from the example above, when we combine the conditions and the

\(^{18}\) These intervals are chosen so that the REE (homogeneous rational expectation equilibrium) values for the market characterized are centered in these intervals.
Fig. 2. Fuzzy sets for the forecast parameter ‘a’.

Fig. 3. Fuzzy sets for the forecast parameters ‘b’.

forecast parameters, we obtain a complete rule which we would code as: [0 1 3 0 2 | 2 4]. In general, we can write a rule as: \[[x_1, x_2, x_3, x_4, x_5 | y_1, y_2]\], where \( x_1, x_2, x_3, x_4, x_5 \in \{0, 1, 2, 3, 4\} \) and \( y_1, y_2 \in \{1, 2, 3, 4\} \). We would therefore interpret the rule \[[x_1, x_2, x_3, x_4, x_5 | y_1, y_2]\] as:

If \( p \times r / d \) is \( x_1 \) and \( p / MA(5) \) is \( x_2 \) and \( p / MA(10) \) is \( x_3 \) and

\( p / MA(100) \) is \( x_4 \) and \( p / MA(500) \) is \( x_5 \), then \( a \) is \( y_1 \) and \( b \) is \( y_2 \)

The example represents the form of the typical rule modeled in the system.
It is important to point out that conventional Boolean logic, unlike fuzzy logic, cannot tolerate inconsistencies. For instance, consider the problem of classifying whether someone is tall or short. Boolean logic requires that one define a crisp cut-off height to separate tall people from short people. So when using Boolean logic, if a person is tall, this person cannot be short at the same time. Fuzzy logic on the other hand allows us to classify a person as belonging to both the fuzzy set of tall and short people. As the reader can see from Figs. 1-3, there is no clear cut-off for a fuzzy set; it is therefore quite conceivable for the above to happen. In fact, it is this flexibility that allows us not only to better describe things but also to do so in an economical fashion.

3.3.5. Fuzzy rule bases as market hypotheses

A genetic-fuzzy classifier contains a set of fuzzy rules that jointly determine what the price expectations should be for a given state of the market. We call a set of rules a rule base. Each rule base represents a tentative hypothesis about the market and reflects a ‘complete’ belief. This point needs further clarification.

Suppose the fuzzy rule is ‘If $p*r/d$ is high then $a$ is low and $b$ is high’. This rule by itself does not make much sense as an hypothesis because it does not specify what the forecast parameters should be for the remaining contingencies, for example the case where $p*r/d$ is ‘low’, ‘moderately low’ or ‘moderately high’. Three additional rules are therefore required to form a complete set of beliefs. For this reason, each rule base contains four fuzzy rules. Fig. 4 shows an example of a rule base, coded as a set of four 7-bit strings. This rule base qualifies as a ‘complete’ belief.

We allow each agent in our model to work in parallel with several distinct rule bases. In order to keep the model manageable yet maintain the spirit of competing rule bases, we allow each agent to work with a total of five rule bases. Therefore, at any given moment, agents may entertain up to five different market hypotheses (rule bases). Hence, it is quite possible that each agent may derive several different price expectations at any given time. To sort out which of these price expectations to rely on, an agent looks at the relative forecast

<table>
<thead>
<tr>
<th>Conditions</th>
<th>Forecast Parameters</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 0 0 2 3</td>
<td>3 2</td>
</tr>
<tr>
<td>4 0 0 1 4</td>
<td>1 3</td>
</tr>
<tr>
<td>2 0 0 4 2</td>
<td>2 4</td>
</tr>
<tr>
<td>3 0 0 3 1</td>
<td>4 1</td>
</tr>
</tbody>
</table>

Fig. 4. A complete and consistent rule base."
accuracy of the rule bases and acts on the one that has recently proven to be the most accurate. This is the same set up used in the SFASM.

In a separate experiment, we deviate from this strictly deterministic choice system and allow agents to sometimes select the rule base to use in a probabilistic manner. The timing of this alternative choice framework is triggered by a random event. Our interest in this modification is motivated by the extensive literature on group polarization, a well-documented phenomenon (for instance, Lamm and Myers, 1978). Shiller (1989, Chapter 2) in a review of group polarization phenomena has suggested that the polarization of negative attitudes by a group may be triggered by specific events. We think of these negative attitudes as ‘doubts’ about whether the perceived best rule base, as measured by its current forecast accuracy, is actually best. We allow such doubts to arise probabilistically, but with a very low probability of occurrence. If the state ‘doubt’ occurs, we tie the probability of selecting a rule base from the five held by an investor at the time, to its relative forecast accuracy so the one which has proven to be the most accurate will have the highest likelihood of being selected. However, when doubt prevails, no rule base will be selected with certainty. In other words, we are assuming that agents will sometimes lose confidence in their best forecasting model, and in those instances they will utilize their second-best or third-best models, etc. This modification is in fact a key extension of the SFASM framework. We show later that the inclusion of this feature produces return kurtosis measures that are more in line with observed data than those produced under the model structure analyzed by LeBaron et al. (1999), while simultaneously generating return and volume behavior that are otherwise similar to actual data.

3.3.6. An example

We now turn to an example of how the fuzzy system operates. Consider a simple fuzzy rule base with the following four rules.

If $0.5p/MA(5)$ is low then $a$ is moderately high and $b$ is moderately high.
If $0.5p/MA(5)$ is moderately low then $a$ is low and $b$ is high.
If $0.5p/MA(5)$ is high then $a$ is moderately low and $b$ is moderately low.
If $0.5p/MA(5)$ is moderately high then $a$ is high and $b$ is low.

Now suppose that the current state in the market is given by $p = 100$, $d = 10$, and $MA(5) = 100$. This gives us, $0.5p/MA(5) = 0.5$. The response of each rule and the resultant fuzzy sets for the two forecast parameters, given this state of the market, are illustrated in Figs. 5–10. In particular, note the responses for the 1st and 3rd rules. In these cases, the membership value for those fuzzy sets representing $0.5p/MA(5)$ is zero since 0.5 is outside their domains, consequently the forecast parameters associated with these rules will also have zero membership values. Thus, only the 2nd and 4th rules contribute to the resultant fuzzy sets for the forecast parameters $a$ and $b$. The resultant information must now be
translated into specific values for $a$ and $b$. We employ the centroid method to accomplish this. This amounts to calculating the centroid of the enclosed areas in Figs. 9 and 10.\(^{20}\) We ‘defuzzify’ these resultant fuzzy sets using the centroid method, obtaining 0.95 and 4.5 for the parameters ‘$a$’ and ‘$b$’, respectively, as shown in Figs. 9 and 10. Substituting these forecast parameters into Eq. (4) gives us the forecast for the next period price and dividend of:

$$E(p + d) = 0.95(100 + 10) + 4.5 = 109.$$  

3.3.7. Forecast accuracy and fitness values

Following LeBaron et al. (1999), we measure forecast accuracy for a rule base by the inverse of $e^2_{t,i,j}$. The variable $e^2_{t,i,j}$ is the moving average of the squared forecast error and is defined as:

$$e^2_{t,i,j} = (1 - \theta) e^2_{t-1,i,j} + \theta[(p_t + d_t) - E_{t-1,i,j}(p_t + d_t)]^2,$$

\(^{20}\)In the centroid method, which is sometimes called the center of area method, the defuzzified value is defined as the value within the range of variable $x$ for which the area under the graph of membership function $M$ is divided into two equal sub-areas (Klir and Yaur, 1995).
Fig. 6. Response of 2nd rule of example rule base to parameter $0.5p/MA(5) = 0.5$.

where $\theta$ is a constant weight, subscript $i$ and $j$ denote the $i$th individual and the $j$th rule base, and $t$ indexes time. In each period the agents refer to $e^2_{t,i,j}$ when deciding which rule base to rely on.

The variable $e^2_{t,i,j}$ is also used for several other purposes. First it is used by agent $i$ as a proxy for the forecast variance $\sigma^2_{t,i,j}$ when setting demand. Second, the value for $e^2_{t,i,j}$ is used to compute what we will label a fitness measure. In the experiments, agents revise their rule bases on average every $k$ periods. Operationally, this means that the GA is invoked every $k$ periods. The fitness measure is used to guide the selection of rule bases for ‘crossover’ and ‘mutation’ in the GA. A GA creates new rule bases by ‘mutating’ the values in the rule base array, or by ‘crossover’ — combining part of one rule base array with the complementary part of another. In general, rule bases that fit the data well, will be more likely to reproduce whereas less fit rule bases will have a higher probability of being eliminated. The fitness measure of a rule base is calculated as follows:

$$f_{t,i,j} = -e^2_{t,i,j} - \beta s.$$  \hspace{1cm} (6)
The parameter $\beta$ is a constant and $s$ is the specificity of the rule base.\textsuperscript{21} The fitness measure imposes higher costs on rules that lead to larger squared forecast errors and which employ greater specificity. Specificity equals the number of information bits that contribute to the conditional part of a rule base. Recall that a bit does not contribute if it is defined with a zero in the conditional part of a rule. For instance, the rule base in Fig. 4 has specificity equal to 12. The parameter $\beta$ is introduced to penalize specificity. The purpose is to discourage agents from carrying bits that are superfluous or redundant, an ingredient that seems natural in a model of reasoning characterized by limits on the ability to store and process information. Thus, the more specific the conditions are in a rule base, the lower its fitness will be, keeping other things constant. The net effect of this is to ensure that a bit is used only if agents genuinely find it useful in predictions and in doing so introduces a weak drift towards configurations containing only zeros.

\textsuperscript{21}A similar specification is employed by LeBaron et al. (1999).
Fig. 8. Response of 4th rule of example rule base to parameter $0.5p/MA(5) = 0.5$.

3.3.8. Recapitulation

The model begins with a dividend, $d_t$, announced publicly at time period $t$. Based on this information, the five market descriptors [$p*r/d$, $p/MA(5)$, $p/MA(10)$, $p/MA(100)$, and $p/MA(500)$] are computed and the forecast models identified. Agents then generate several different price expectations using their genetic-fuzzy classifiers. They forecast next period's price and dividend ($E_{t,t+1}[p_{t+1} + d_{t+1}]$) by using the forecast parameters from the rule base that has proven to be the most accurate recently. With this expectation and its variance, they use Eq. (2) to calculate their desired share holdings. The market clearing price is then determined at $t$ as if by a tatonement process. Once the market clears, the price and dividend at time $t$ are revealed and the accuracies of the rule bases are updated.

Learning in the model happens at two different levels. On the surface, learning happens rapidly as agents experiment with different rule bases and over time

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22 We mentioned earlier in Section 3.3.5 that a separate experiment involved the introduction of a state of 'doubt' in which the rule base choice is probabilistic versus deterministic as described here.
discover which rule bases are accurate and worth acting upon and which should be ignored. At a deeper level, learning occurs at a slower pace as the GA discards unreliable rule bases to make room for new ones through crossover and mutation. The new, untested rule bases that are created will not cause disruptions because they will be acted upon only if they prove to be accurate. This avoids brittleness and provides what machine-learning theorists call ‘gracefulness’ in the learning process.

3.4. Experiments

This section describes the controlled experiments we have conducted. We examine the implications of our conjecture about the reasoning process for the
behavior of prices and volume in a dynamic setting. In these experiments, we kept almost all of the model’s parameters the same so that comparisons can be made of the market outcomes under identical conditions with only controlled changes. The primary control parameter, as in LeBaron et al. (1999), is the learning frequency constant, \( k \).

Learning frequency refers to the frequency at which a GA is invoked in the model. When the learning frequency is high, a GA is invoked more frequently and agents will revise their rule bases more often. In contrast, when the learning frequency is low, a GA is invoked less often, so agents will revise their rule bases at a slower pace. Recall that agents are not able to use deductive reasoning to shape their price expectations. Instead, they use inductive reasoning which basically amounts to formulating tentative hypotheses and testing these hypotheses repeatedly against observed data. Under such a scheme, it is clear that the learning frequency will play a key role in determining the structure of the rule bases and how well agents are able to coordinate their price expectations. When learning frequency is high, agents will revise their beliefs frequently so they will typically not have adequate time to fully explore whether their market hypotheses are consistent with those belonging to other agents. At the same time, if agents revise their hypotheses frequently, their hypotheses are more likely to be influenced by transient behavior in the time series of market variables. These factors together will make it difficult for agents to converge on an equilibrium price expectation. In contrast, when learning frequency is low, agents will have more time between revising their rule bases to explore their hypotheses. Furthermore, their hypotheses will also tend to be based on longer horizon features in the time series of market variables. Consequently, agents are more likely to converge on an equilibrium price expectation.

The model’s parameters that are common to all the experiments are tabulated in Table 1. We conducted three sets of experiments. In the first and second experiments, the learning frequency, \( k \), is equal to 200 and 1000, respectively. This means that the agents learn on average once every 200 time periods in the first experiment and once every 1000 periods in the second experiment. In both these experiments, we assume that the agents always form their forecasts using the most accurate rule base.

In the third experiment, we set the learning frequency, \( k \), equal to 200, and we let the agents select the rule base to act upon in a probabilistic manner when formulating their forecasts. In the spirit of our earlier discussion, we view the event that triggers this modification in the choice system to be associated with a state of ‘doubt’. But the probability that such an event occurs is intentionally kept very small. We assume that there is a 0.1% probability that an agent will decide to select the rule base to act upon in a probabilistic fashion. However,
Table 1
Parameter values

Common parameter values in the simulations of an artificial stock market in which a risky security is traded and investors are expected utility of final wealth maximizers. Investors represented in the model employ induction and reason as if by fuzzy logic when forming their expectations. The process is formally modeled as a genetic-fuzzy classifier system.

<table>
<thead>
<tr>
<th>Description of parameters</th>
<th>Values</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mean dividend (( \bar{d} ) in Eq. (1))</td>
<td>10</td>
</tr>
<tr>
<td>Autoregressive parameter (( \rho ) in Eq. (1))</td>
<td>0.95</td>
</tr>
<tr>
<td>Variance (( \sigma^2 ) in Eq. (1))</td>
<td>0.0743</td>
</tr>
<tr>
<td>Risk-free interest rate (( r ))</td>
<td>0.1</td>
</tr>
<tr>
<td>Risk aversion parameter (( \lambda ) in Eq. (2))</td>
<td>0.5</td>
</tr>
<tr>
<td>Weight (( \theta ) in Eq. (5))</td>
<td>1/75</td>
</tr>
<tr>
<td>Cost of specificity (( \beta ) in Eq. (6))</td>
<td>0.06</td>
</tr>
<tr>
<td>Number of bits for market descriptors</td>
<td>5</td>
</tr>
<tr>
<td>Number of bits for forecast parameters</td>
<td>2</td>
</tr>
<tr>
<td>Crossover probability</td>
<td>0.1</td>
</tr>
<tr>
<td>Mutation probability*</td>
<td>0.9</td>
</tr>
<tr>
<td>Number of agents (also Number of shares)</td>
<td>25</td>
</tr>
<tr>
<td>Number of rule bases per agent</td>
<td>5</td>
</tr>
<tr>
<td>Number of fuzzy rules per rule base</td>
<td>4</td>
</tr>
</tbody>
</table>

*This is the probability that an agent will have one of his rule bases subjected to mutation. When a particular rule base is selected for mutation, the probability that each bit is mutated is 0.03, and the probability that a bit will be transformed from 0's to non-0's or vice versa is 0.5.

when one agent decides to do so, we assume that all the remaining agents follow his example, in other words, a polarization of attitudes arises. In other words, we are assuming that if there is an event that shakes the confidence of one agent, then the same event will also shake the confidence of all other agents. The probability that agent \( i \) will select his rule base \( j (j = 1, 2, \ldots, 5) \) is then linked to the relative forecast accuracy of the rule base. Specifically, when a state of doubt arises, we rank the five rule bases that an agent carries from 0 to 2, in increments of 0.5, and compute the selection probability for each rule base as follows:

\[
P_j = \frac{\text{ranking}_j}{\sum_{i=1}^{5} \text{ranking}_i}
\]

where \( j \) indexes the \( j \)th rule base.\(^{24}\) Note that agents do not decide to choose a ‘random’ rule set. They do however, all decide at the same time to deviate from

\(^{24}\) Recognize that while a polarization of attitudes can arise with respect to doubt about which of their five rule bases is best, the framework we analyze would assign the same five rule bases to any two agents only by chance.
the best rule set with some probability when the state of ‘doubt’ occurs. In addition the probability that governs which rule base will be chosen when the above event happens may be unique for each agent. This probability is determined by the ratings each agents has assigned to each of his five rule bases.

We follow LeBaron et al. (1999) in referring to the two cases \( k = 200 \) and \( 1000 \) as ‘fast learning’ and ‘slow learning’. Learning takes place asynchronously. In other words, not all the agents in the model update their rule bases simultaneously.

We began with a random initial configuration of rules. We then simulated the market experiment for 100,000 periods to allow any asymptotic behavior to emerge. Subsequently, starting with the configuration attained at \( t = 100,000 \) we simulated an additional 10,000 periods to generate data for the statistical analysis discussed in the next section. We repeated the simulations 10 times under different random seeds to facilitate the analysis of regularities that emerge from repeated observation of a complete market realization.

4. Results

Simulation results from our experiments show that the model is able to generate behaviors that bear a strong resemblance to many of the regularities that have been observed in real financial markets. We discuss these results in the following sections. Throughout our discussion, the mnemonic REE will stand for Rational Expectations Equilibrium.

4.1. Asset prices and returns

Figs. 11 (Experiment 1), 12 (Experiment 2) and 13 (Experiment 3) present snapshots of observed price behavior over typical windows. These graphs present the price series over a shortened window, so that the visual relation between the market price and the REE price is not obscured by the compression necessary when graphing the entire history. An interesting feature that stands out in these three graphs is that the market price is consistently below the REE price. Unlike the situation in a Rational Expectations Equilibrium, in our model agents are not able to coordinate their forecasts perfectly, and this causes the market price to be more volatile than the REE price. Inspection of the REE price solution, under the conjecture that prices are linear in dividends, reveals why the market price of the model is below the REE price (see the Appendix). As the appendix shows, increased price variability reduces the market clearing price, ceteris paribus, because the expression for the intercept parameter in what now is the temporary market clearing price equation, falls with increased variability. The figures also show that the difference between the market price and the REE price is larger for the two fast learning cases \( k = 200 \). Again, this is a result of high price variability.
Further evidence on the relation between the price level and price variability appears in Table 2. We compute the mean and standard deviation for the market price across the 10 complete market realizations for the three sets of experiments and present these results in the first two rows of Table 2. We
estimate the standard deviation for the REE price to be equal to 5.4409. Judging from the standard deviations reported in Table 2, it is clear that the market price in all three sets of experiment is more volatile than the REE price, and that the mean price falls as variability rises.\textsuperscript{25} The larger volatility in the fast learning case ($k = 200$) can be attributed to the more frequent revision of rules by agents. The rules in the fast learning case are also more likely to be based on transient shorter horizon features in the time series of market variables. Agents therefore find it necessary to employ different rule bases to form their expectations at different times. This regular switching in the agents’ beliefs, in turn, can give rise to higher volatility because they need time to adapt to the changes.

The remaining rows in Table 2, except the last, present results on the behavior of the residual series ($\varepsilon_t$) obtained from the following regression equation:

$$p_{t+1} + d_{t+1} = a + b(p_t + d_t) + \varepsilon_{t+1}.$$  \hspace{1cm} (8)

We know that in a homogeneous REE, the residual series should be independent and identically distributed as $N(0, 4)$, given our assumptions.\textsuperscript{26} This means

\textsuperscript{25} For evidence on volatility of market price and related tests, see Leroy and Porter (1981a, b) and Shiller (1981). Shiller (1988) is a discussion of the volatility debate.

\textsuperscript{26} Note that $\sigma^2_{p+d} = (1 + f)^2 \sigma^2_p = [\rho/(1 + r - \rho)]^2 \sigma^2_e$. This result is derived in Appendix A. This equation will give us a variance of 4 for the parameter values listed in Table 1.
Table 2
Summary statistics for prices, model residuals and implied returns computed from simulations of an artificial stock market in which agents form expectations based upon a genetic-fuzzy classifier system. Experiment 1 involves slow learning. Experiments 2 and 3 involve fast learning.

<table>
<thead>
<tr>
<th>Variables</th>
<th>Experiment 1 (slow learning, $k = 1000$)</th>
<th>Experiment 2 (fast learning, $k = 200$)</th>
<th>Experiment 3 (fast learning with doubt, $k = 200$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Panel A. Price ($P_t$)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Mean ($\bar{P}$)</td>
<td>76.0155</td>
<td>73.5214</td>
<td>69.5040</td>
</tr>
<tr>
<td>Std. Dev. ($\sigma_P$)</td>
<td>6.0967</td>
<td>6.6333</td>
<td>7.0422</td>
</tr>
<tr>
<td>Panel B. Residual ($e_t$)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Standard deviation</td>
<td>2.0764 (0.0663)</td>
<td>2.0860 (0.0355)</td>
<td>2.2777 (0.1025)</td>
</tr>
<tr>
<td>Skewness</td>
<td>$-0.0038$ (0.0242)</td>
<td>$-0.0100$ (0.0362)</td>
<td>$-0.046$ (0.1078)</td>
</tr>
<tr>
<td>Kurtosis</td>
<td>3.0862 (0.0469)</td>
<td>3.0742 (0.0590)</td>
<td>5.5796 (1.0777)</td>
</tr>
<tr>
<td>$\rho_1(e_t)$</td>
<td>0.0102 (0.0551)</td>
<td>0.0271 (0.0589)</td>
<td>$-0.1391$ (0.0903)</td>
</tr>
<tr>
<td>$\rho_1(e_t^2)$</td>
<td>0.0351 (0.0177)</td>
<td>0.0427 (0.0149)</td>
<td>0.324 (0.0874)</td>
</tr>
<tr>
<td>Panel C. Returns ($R_t$)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Standard deviation</td>
<td>0.0277 (0.0013)</td>
<td>0.0295 (0.0021)</td>
<td>0.0367 (0.0037)</td>
</tr>
<tr>
<td>Skewness</td>
<td>0.0721 (0.0226)</td>
<td>0.2841 (0.5634)</td>
<td>0.6470 (0.3426)</td>
</tr>
<tr>
<td>Kurtosis</td>
<td>3.1715 (0.0399)</td>
<td>4.5663 (3.549)</td>
<td>8.3689 (2.952)</td>
</tr>
<tr>
<td>$\rho_1(R_t)$</td>
<td>$-0.0095$ (0.0726)</td>
<td>0.0734 (0.0941)</td>
<td>$-0.1038$ (0.0737)</td>
</tr>
<tr>
<td>$\rho_1(R_t^2)$</td>
<td>$-0.0058$ (0.0703)</td>
<td>0.0948 (0.1416)</td>
<td>$-0.0685$ (0.0641)</td>
</tr>
</tbody>
</table>

*a* This table gives the average of each variable over the 10 simulations conducted for each experiment. The numbers in parentheses are the standard errors. Skewness, kurtosis and first-order autocorrelation are computed in the conventional manner (see Greene, 1993).

*b* Residuals are computed from the model $p_{t+1} + d_{t+1} = a + b(p_t + d_t) + e_{t+1}$.

*c* The numbers for the ARCH LM(1) tests are the means of the $F$-statistics for the 10 simulations for each experiment and the numbers in square brackets are the percentage of the number of tests that reject the null hypothesis of ‘no ARCH’.

*d* Returns are computed as the change in price from $t$ to $t+1$ plus the dividend at $t+1$ all divided by the price at $t$. 
Table 3
Summary statistics for the daily with-dividend returns for the common stocks of the Walt Disney Company, Exxon Corporation and IBM Corporation. Data are from the period 01/03/68 through 12/31/98.

<table>
<thead>
<tr>
<th>Variables</th>
<th>Disney</th>
<th>Exxon</th>
<th>IBM</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mean</td>
<td>0.0009</td>
<td>0.0006</td>
<td>0.0005</td>
</tr>
<tr>
<td>Standard deviation</td>
<td>0.0200</td>
<td>0.0127</td>
<td>0.0149</td>
</tr>
<tr>
<td>Skewness</td>
<td>-0.2310</td>
<td>-0.2870</td>
<td>-0.0104</td>
</tr>
<tr>
<td>Kurtosis</td>
<td>13.0960</td>
<td>22.5141</td>
<td>14.1120</td>
</tr>
<tr>
<td>$\rho_1(R_t)$</td>
<td>0.072c</td>
<td>0.015</td>
<td>0.001</td>
</tr>
<tr>
<td>$\rho_1(R_t^2)$</td>
<td>0.154c</td>
<td>0.489c</td>
<td>0.181c</td>
</tr>
</tbody>
</table>

Panel A. Daily returns from 01/03/63 to 12/31/98

<table>
<thead>
<tr>
<th>Variables</th>
<th>Disney</th>
<th>Exxon</th>
<th>IBM</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mean</td>
<td>0.0009</td>
<td>0.0006</td>
<td>0.0005</td>
</tr>
<tr>
<td>Standard deviation</td>
<td>0.0196</td>
<td>0.0122</td>
<td>0.0146</td>
</tr>
<tr>
<td>Skewness</td>
<td>0.0119</td>
<td>0.0486</td>
<td>0.3666</td>
</tr>
<tr>
<td>Kurtosis</td>
<td>7.6436</td>
<td>5.1662</td>
<td>8.0008</td>
</tr>
<tr>
<td>$\rho_1(R_t)$</td>
<td>0.077c</td>
<td>0.023b</td>
<td>0.007</td>
</tr>
<tr>
<td>$\rho_1(R_t^2)$</td>
<td>0.132c</td>
<td>0.134c</td>
<td>0.127c</td>
</tr>
</tbody>
</table>

Panel B. Daily returns from 01/03/63 to 12/31/98 (excluding the observations from Oct. 19, 1987 through Oct 21, 1987)

*Skewness, kurtosis and first-order autocorrelation are computed in the conventional manner (see Greene, 1993).

Denotes significance at 5% confidence level.

dDenotes significance at 1% confidence level.

that the theoretical standard deviation for $\hat{\varepsilon}_t$ should be 2. Furthermore, under a Gaussian distribution the excess kurtosis of the error should be zero. We compare these theoretical results to those in the third and fifth rows in Table 2. It is apparent from the values in the third row that the residual series from Experiment 3 exhibits more variability than what is implied by the REE solution. The standard deviation of the residual series from Experiments 1 and 2 are slightly larger than 2.

The fifth row shows that the residual series from Experiments 1 and 2 exhibit slightly excess kurtosis. But, the kurtosis from Experiment 3 is much larger than these two and the magnitude is consistent with that of the daily common stock returns presented in Table 3. Table 3 presents summary statistics for the daily with-dividend returns of three common stocks: Disney, Exxon, and IBM. The data span the period 1/03/63–12/31/98 and are from the CRSP Daily Stock Return File. Our model is capable of creating market crashes like what we have seen in actual financial markets. We have excluded those simulations that contain market crashes in the computation of the descriptive statistics shown in
When we included those simulations that contain market crashes in the summary calculations, the mean kurtosis became 24.7239 which is consistent with the result in the top panel in Table 3.\footnote{When we included those simulations that contain market crashes in the summary calculations, the mean kurtosis became 24.7239 which is consistent with the result in the top panel in Table 3.} As mentioned earlier, we began each simulation with a random initial configuration of rule bases. What we observed during the course of each simulation was that kurtosis is always rather large during the first few hundred time steps (usually up to roughly 1000 time steps) during which the agents’ learning curves are the steepest. The agents in the model have the unenviable task during the early phase of the simulation of figuring out how to coordinate their expectations by sorting out the rules that work from the many arbitrary rules they have been endowed with. However, once the agents have identified rule bases that seem to work well, excess kurtosis decreases rapidly until it is almost equal to zero. From that point on, it is extremely difficult (but not impossible) for the model to generate further excess kurtosis without some exogenous perturbation. We may observe spurts of excess kurtosis arising from a substantial sudden change in the mind of the agents as a whole, such as in the case of an emerging trend, although this is rare. In such a case, the excess kurtosis usually lasts for a short time and such infrequent events are easily washed out when we consider a long return series, except in the presence of a large price deviation.

We conclude that the reason it is so difficult to generate large kurtosis (without relying on an exogenous perturbation) is because it is difficult to break the coordination among the agents in the model once they have established some form of mutual understanding. This is in fact consistent with received wisdom. In actual stock markets, price swings are often triggered by exogenous events (such as rumors or earnings surprises) which are then perpetuated by endogenous interactions among traders. We suspect the large kurtosis observed in actual returns series may have originated from such exogenous events. It is in this spirit that we introduced what we earlier referred to as a state of ‘doubt’.

The sixth row in Table 2 presents statistics on the autocorrelation in the residuals from Eq. (8). The autocorrelation coefficients tell us whether there is any linear structure remaining in the residuals. Other than the result for Experiment 3, the results for the first two experiments show that there is little autocorrelation in the residuals. This corresponds to the low actual autocorrelations shown in Table 3.

Several authors have shown that security returns exhibit conditional time-varying variability (for instance, Engle, 1982; Bollerslev, 1986; Bollerslev et al.,...
1992; Glosten et al., 1994; Nelson, 1991). We test for ARCH dependence in the residuals and present the results in Rows 7 and 8 of Table 2. We test for ARCH effects in two different ways. In Row 7, we present the first order autocorrelation of the squared residuals. In Row 8, we perform the ARCH LM test proposed by Engle (1982). Both of these tests reveal that there is ARCH dependence in the residuals. However, the effect is more pronounced for the two fast learning cases. In these two experiments we rejected the null hypothesis of ‘no ARCH’ at the 95% confidence level for all 20 simulations using the ARCH LM tests. The null was rejected for only 6 of the simulations in the slow learning case.

We also computed statistics for the returns based upon the simulated prices and dividends. These statistics appear as the final five rows of Table 2. Now comparing Tables 2 and 3 we see that the correspondence between the returns computed from the simulated data, and the actual return data are even more striking. The standard deviation, skewness as well as kurtosis statistics for the simulated returns from experiment 3 are very similar to those shown for the stocks in Table 3.

4.2. Trading volume

Figs. 14 and 15 present snapshots of observed trading volume over a typical window for both the fast-learning and slow-learning cases. Trading is active, consistent with real markets. In fact, we find that the volume of trade in Experiment 3, on occasion, can exceed 50% of the total number of shares available in the market. In both Experiments 1 and 2, the volume of trade can be as high as 33%.²⁸ The summary statistics for trading volume are presented in Table 4.

Figs. 16–18 plot the volume autocorrelations for both the fast-learning and slow-learning experiments. In these three plots, the broken lines are one standard deviation away from the continuous line, which is the mean of the autocorrelations computed for each of the 10 experiments for each case. These plots show that trading volume is autocorrelated. This result lines up well with the positive autocorrelations usually found in time series of the volume traded for common stocks.²⁹ Fig. 19 shows a plot of the daily volume autocorrelations for Disney, Exxon, and IBM. Figs. 16–18 show that the model produces volume autocorrelation behavior strikingly similar to what is observed for the actual data presented in Fig. 19.

Figs. 20 and 21 present the cross-correlation between volume traded and volatility. In Fig. 20, the cross-correlation is between volume traded and squared

²⁸ Unlike LeBaron et al. (1999), we have not placed an upper bound on the number of shares that can be traded by any single agent, in particular we impose no short sale restriction.

²⁹ See Karpov (1987).
returns using data from Experiment 3. The broken lines are one standard deviation away from the continuous line, which is the mean of the 10 simulations for Experiment 3. In Fig. 21, which shows the results for Disney, Exxon and IBM, we used the squared returns as a proxy for volatility and compute the
Table 4
Summary statistics for volume of trading per time step computed from simulations of an artificial stock market in which agents form expectations based upon a genetic-fuzzy classifier system. Experiment 1 involves slow learning. Experiments 2 and 3 involve fast learning. Summary statistics are based upon 10 simulations of each experiment.

<table>
<thead>
<tr>
<th>Variables</th>
<th>Experiment 1 (slow learning, $k = 1000$)</th>
<th>Experiment 2 (fast learning, $k = 200$)</th>
<th>Experiment 3 (fast learning with doubt, $k = 200$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mean $^a$</td>
<td>0.7346</td>
<td>0.8196</td>
<td>0.9369</td>
</tr>
<tr>
<td>Maximum $^b$</td>
<td>7.2514</td>
<td>8.2833</td>
<td>12.5978</td>
</tr>
<tr>
<td>Minimum $^b$</td>
<td>0.0225</td>
<td>0.0322</td>
<td>0.0203</td>
</tr>
</tbody>
</table>

$^a$Grand mean per time step over all simulations for a given experiment.
$^b$Maximum (Minimum) per time step over all simulations for a given experiment.

Fig. 16. Experiment 1: slow learning, $k = 1000$. Volume autocorrelations measured at various lags.

cross correlation between this variable and the volume traded. The plots in Fig. 20 show that volume is contemporaneously correlated with volatility in the simulations, and again, these results are strikingly similar to those for actual securities as presented in Fig. 21. Generally speaking as well, the volume and volatility results for the market modeled in this paper are quite similar to those found by LeBaron et al. (1999) despite the significant difference between the two models with respect to the inductive process ascribed to economic agents.
Fig. 17. Experiment 2: fast learning, $k = 200$. Volume autocorrelations measured at various lags.

Fig. 18. Experiment 3: fast learning with doubt, $k = 200$. Volume autocorrelations measured at various lags.
Fig. 19. Sample daily volume autocorrelations for the common stocks of Disney, Exxon and IBM.

Fig. 20. Experiment 3: fast learning with doubt, $k = 200$. Crosscorrelation of squared returns at $(t + j)$ with volume at $t$. 
Fig. 21. Sample crosscorrelations between squared daily returns at \((t + j)\) and volume at \(t\) for the common stocks of Disney, Exxon and IBM.

4.3. Market efficiency

Fig. 22 plots a snapshot of the difference between the REE price and the market price over a typical window. This plot displays periods during which the market price is highly correlated with the REE price.\(^{30}\) However, there are periods of sporadic wild fluctuations during which this relation is broken. We pointed out earlier that the market prices of our model are expected to be less than the price implied by a rational expectations equilibrium because the market price exhibits greater volatility. However, if the simulated price tracks the REE price after adjusting for volatility, we should expect to see a constant difference between the two prices. The differences plotted in Fig. 22 are not constant across time, implying that the expectations held by the agents represented in the model are not always consistent with a rational market equilibrium. We conclude from these results that the market moves into and out of various states of efficiency. The figure does suggest, however, that market prices have some tendency to return to a constant distance from the REE price after departures occur.

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\(^{30}\) This result lends support to our assumption that Eq. (2) describes individual demands. Despite the fact that prices during periods that track the REE solutions are not normally distributed, market clearing prices based upon demands as given by (2) give the correct values after accounting for the additional variability in the system due to learning.
This type of behavior is not uncommon in real financial markets. The historical record suggests that most of the time, prices of financial securities appear to be set in an efficient fashion. However, there have been occasions during which prices depart and appear to exhibit a behavior unrelated to fundamentals. This is especially evident during those periods that most observers would classify as either bubbles or crashes. A good example is the change that occurred on October 27, 1997. On that day the Dow Jones Industrial Average dropped 554.26 points. The peculiar fact is not the swing itself, but the fact that the fundamentals of the US economy were perceived to be strong at the time, including the prospects for continued growth, low inflation and low unemployment. Just as peculiar is that on the following day, October 28, 1997, the DJIA posted a 337 point gain amid record volume. The movement into and out of periods during which prices are set efficiently, is, as we have shown, entirely consistent with a world in which the type of fuzzy reasoning and induction modeled in this paper prevail.

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51 The DJIA closed at 7715.41 on Friday, October 24, 1997. The close on Monday, October 27, 1997 was 7161.15. This was in turn followed by it closing on Tuesday, October 28 at 7498.32. This sequence of outcomes was not peculiar to the DJIA as most other major market indices also experience major changes.
5. Conclusions

This paper argues that many of the regularities observed in common stock returns can be explained by allowing agents in otherwise traditional asset market models to form their expectations in a manner akin to how investors would form their expectations in real life. In particular, because the environment that investors operate in is ill-defined, they will have to rely on their innate abilities to analyze in fuzzy terms and reason inductively. We show that these traits can be faithfully captured by a genetic-fuzzy classifier system. We subsequently assert that models, which endow agents with such a reasoning process, will account for some of the documented empirical puzzles observed in real markets and will generate price and return behavior consistent with actual data.

We document a close correspondence between statistics computed from our simulations and statistics computed from actual return series. These include descriptive statistics for returns as well as the resulting behavior of volume autocorrelations and volume and volatility cross-correlations. We show that a basic framework, however, is not capable of generating return kurtosis measures that are consistent with actual data. A modification of the model to allow for the intrusion of a (low probability) state of ‘doubt’ about what would otherwise have been identified as the best prediction rule, is shown to produce return kurtosis measures that are more in line with actual data.

Summing up, our model can account simultaneously for several regularities observed in real markets, that have been a struggle to rationalize within the context of the traditional rational expectations paradigm. First, our model can give rise to active trading. Second, our model supports the views of both academicians and market traders. Academic theorists in general view the market as rational and efficient. Market traders typically see the market as psychological and imperfectly efficient. In our model, we find that the market moves in and out of various states of efficiency. Furthermore, we find that when learning occurs slowly, the market can approach the efficiency of a REE. Finally, descriptive statistics for the returns implied by the simulated price and dividend series are shown to be consistent with results for actual data, as are the results for volume and volatility.

Finally, our work is very much like the work of LeBaron et al. (1999) who present results for a model similar in nature to the one examined herein, with similar results. The two models however exhibit one important difference with respect to the reasoning process ascribed to economic agents. We have argued that the ability of agents to process extensive amounts of data internally is limited, and that existing evidence from other social sciences suggests that individuals reason as if by the precepts of fuzzy logic. While LeBaron et al. also model reasoning as an induction process, their model assigns to each agent a total of 100 rules that are applied in the mapping of market states into price expectations. In contrast we assume that agents reason as if by the use of fuzzy...
logic. Our agents therefore work with only a handful of rule bases, but these have the special characteristic that they are fuzzy rules. Our agents still modify their rules using induction (modeled by a genetic learning system), but accomplish the same objectives as the model of LeBaron et al. with a much smaller set of rules. The fact that a model based upon a fuzzy system generates market behavior similar to a model based upon crisp but numerous rules is appealing because fuzzy decision making has appeal as a reasonable attribute for individuals.

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Appendix A. Solution for a linear homogeneous rational expectations equilibrium of the model

The dividend process and the demand for the security as delineated in the text are given by

\[ d_t = \bar{d} + \rho(d_{t-1} - \bar{d}) + \varepsilon_t, \]

\[ x_{it} = \frac{E_{it}[ P_{t+1} + d_{t+1} ] - p_t(1 + r)}{\lambda \sigma^2_{p+1,t+1}}. \]

We assume that agents conjecture that price is a linear function of the dividend, that is,

\[ p_t = f d_t + e. \]

This allows us to write the conditional expectation and conditional variance of \( (p_{t+1} + d_{t+1}) \) as

\[ E_{i,t}[ p_{t+1} + d_{t+1} ] = E_{i,t}[ (1 + f) d_{t+1} + e ] \]

\[ = E_{i,t}[ (1 + f)(\bar{d} + \rho(d_t - \bar{d}) + \varepsilon_{t+1}) + e ] \]

\[ = (1 + f)(\bar{d} + \rho(d_t - \bar{d})) + e, \]

\[ Var_{i,t}[ p_{t+1} + d_{t+1} ] = Var_{i,t}[ (1 + f) d_{t+1} + e ] \]

\[ = Var_{i,t}[ (1 + f)(\bar{d} + \rho(d_t - \bar{d}) + \varepsilon_{t+1}) + e ] \]

\[ \sigma^2_{p+1} = Var_{i,t}[ (1 + f)\varepsilon ] = (1 + f)^2 \sigma^2_{\varepsilon}. \]
In equilibrium, each agent must hold the same number of shares (since all the agents are equally risk averse). Given that the total number of shares is equal to the total number of agents, each agent must hold only one share at all times in equilibrium. This allows us to set the demand equation to one. We can then substitute the above expression for the one-period ahead forecast into the demand equation to obtain,

$$1 = \frac{(1 + f)(\bar{d} + \rho(d_t - \bar{d})) + e - (fd_t + e)(1 + r)}{\lambda \sigma^2_{p+d}}.$$

The LHS of this expression is a constant. Therefore, the RHS cannot exhibit any dependence on time. So terms containing $d_t$ must vanish. This leads to

$$(1 + f)\rho - (1 + r)f = 0,$$

$$f = \frac{\rho}{(1 + r - \rho)}.$$

The quantity ‘e’ can be obtained by substituting $f$ back into the demand equation and setting demand equal to 1, yielding

$$e = \frac{\bar{d}(f + 1)(1 - \rho) - \lambda \sigma^2_{p+d}}{r}.$$

The relationship between the forecast parameters ‘a’ and ‘b’ in our model and the REE parameters can now be established. We first write the one-period ahead optimal forecast for price and dividend as

$$E(p_{t+1} + d_{t+1}) = \rho(p_t + d_t) + (1 - \rho)[(1 + f)\bar{d} + e].$$

Comparing this equation to $E(p_{t+1} + d_{t+1}) = a(p_t + d_t) + b$, gives

$$a = \rho,$$

$$b = (1 - \rho)[(1 + f)\bar{d} + e].$$

References


