Option Pricing with Genetic Algorithms: Separating Out-of-the-Money from In-the-Money

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Abstract

By separating the case *out-of-the-money* from the case *in-the-money*, this paper extends the study of Chen and Lee (1997) in the application of *genetic algorithms* to option pricing. The boundary condition for the call price in terms of the expiration date is also carefully formulated. With this modification, the GA's performance is improved in the *out-of-the-money* case, more precisely, the *deep* out-of-the-money case.

1 Motivation

Recent applications of artificial neural networks, genetic algorithms and genetic programming to option pricing reveal an asymmetric result, namely, in terms of the *absolute percentage error*, these tools generally performs worse in the *out-of-the-money* case than in the *in-the-money* case.[1] This result may lead to the suggestion that the out-of-the-money case should be separated from the in-the-money case when we apply those tools. This separation can also be motivated by a geometric inspection of the call price curve at its expiration date. At the expiration data, the call price curve has a kink at the place where the stock price (S) equals to the strike price (E). Therefore, one can expect that the curvature of the call price curve can change abruptly at $S = E$.[2] Taking this abrupt change into account, we may improve the GA's performance by separating the *out-of-the-money* case from the *in-the-money* case.

The rest of the paper is organized as follows. In Section 2, we give a brief review on the mathematical model of option pricing. Section 3 will show how GAs can be applied to option pricing. Section 4 presents the experiment design and the simulation results. The results are also compared with those in Chen and Lee (1997). Section 5 leaves the concluding remarks.

2 The Mathematics of Option Pricing

For general readers' convince, a brief review of the mathematics of the Black-Schools option pricing is provided here. For details, the interested reader is referred to Chen and Lee (1997). Black and Scholes (1973) was the first to provide a closed-form solution for the valuation of *European options*. The Black-Scholes option pricing model is based on the principle known as the *no-arbitrage condition* in economics. Let Q_S denote the number of shares of a stock, S the price per share, and Q_C the quantity of calls and C the price per call, then V_H, the value of the hedge portfolio, is simply,

$$V_H = SQ_S + CQ_C. \quad (1)$$

The change in the value of the hedge portfolio is the total derivative of Equation (1)

$$dV_H = Q_S dS + Q_C dC. \quad (2)$$

[1] For example, see Hutchinson, et al. (1994), Figure 5-c; Barucci, et al. (1996), Figure 1; Trigueros (1997), Table 10; Chen and Lee (1997), Figures 3 and 4.

[2] The general reader is referred to Wilmott et al. (1995). The picture mentioned above can be found in the figures 3.6-3.7 of this book.

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We assume that the stock price follows a *geometric Brownian motion process*, i.e., its rate of return can be described as

$$\frac{dS}{S} = \mu dt + \sigma dz \tag{3}$$

where μ is the instantaneous expected rate of return (*drift*), σ the instantaneous standard deviation of the rate of return (*volatility*), dt denotes a small increment of time, and dz is a Wiener process. Since the option's price is a function of the stock's price, its movement over time must be related to the stock's movement over time. To make this relation explicit, we shall, sometimes, use the notation $C(S,\tau)$ to denote the price of the call, where τ is time to maturity. Employing *Ito's Lemma*, $C(S,\tau)$ can be expressed as the following stochastic differential equation:

$$dC = \frac{\partial C}{\partial S}dS + \frac{\partial C}{\partial t}dt + \frac{1}{2}\frac{\partial^2 C}{\partial S^2}\sigma^2 S^2 dt \tag{4}$$

Replacing dC in Equation (2) with the RHS of Equation (4), we can rewrite Equation (2) as follows.

$$dV_H = Q_s dS[\frac{\partial C}{\partial S}dS + \frac{\partial C}{\partial t}dt] + \frac{1}{2}\frac{\partial^2 C}{\partial S^2}\sigma^2 S^2 dt \tag{5}$$

One of the most important insights revealed by Black-Scholes option pricing model is that it can be used as a *hedging vehicle*, i.e., it is possible to continuously adjust the hedge portfolio, V_H, so that it becomes *risk free*. More precisely, the relation in Equation (6) should sustain in the riskless situation.

$$dV_H = Q_s dS + Q_c dC = 0 \tag{6}$$

Without loss of generality, we can normalize Equation (6) by setting $Q_S = 1$ and derive Equation (7) from (6).

$$Q_C = -Q_S\frac{dS}{dC} = -\frac{dS}{dC} \tag{7}$$

The risk-free hedge portfolio will earn the *risk-free rate* in equilibrium if *capital markets are efficient* and the equilibrium relationship is expressed as Equation (8).

$$\frac{dV_H}{V_H} = r_f dt \tag{8}$$

Substituting Equations (8) and (7) into Equation (5), we obtain

$$\begin{aligned} dV_H &= r_f V_H dt \\ &= dS - \frac{\partial S}{\partial C}[\frac{\partial C}{\partial S}dS + \frac{\partial C}{\partial t}dt + \frac{1}{2}\frac{\partial^2 C}{\partial S^2}\sigma^2 S^2 dt] \end{aligned} \tag{9}$$

Equation (9) can be rearranged as follows.

$$\frac{\partial C}{\partial t} = r_f V_H \frac{-\partial C}{\partial S} - \frac{1}{2}\frac{\partial^2 C}{\partial S^2}\sigma^2 S^2 \tag{10}$$

Substituting equation (1) for V_H ,we have

$$\begin{aligned} \frac{\partial C}{\partial t} &= r_f(SQ_S + CQ_C)(-\frac{\partial C}{\partial S}) - \frac{1}{2}\frac{\partial^2 C}{\partial S^2}\sigma^2 S^2 \\ &= r_f C - r_f S\frac{\partial C}{\partial S} - \frac{1}{2}\frac{\partial^2 C}{\partial S^2}\sigma^2 S^2 \end{aligned} \tag{11}$$

Equation (11) is the famous *Black-Scholes partial differential equation*. This partial differential equation can be solved with the following two boundary conditions:

$$C(S,0) = max(S - E, 0) \tag{12}$$

and

$$C(S = 0, \tau) = 0 \tag{13}$$

Black-Scholes(1973) transforms the equation into the heat exchange equation from physics to find the following solution:

$$C = SN(d_1) - Ee^{-r_f\tau}N(d_2) \tag{14}$$

where $d_1 = \frac{\ln(S/E)+r_f\tau}{\sigma\sqrt{\tau}} + \frac{1}{2}\sigma\sqrt{\tau}$, $d_2 = d_1 - \sigma\sqrt{\tau}$, and $N(d)$ is the cumulative distribution function for the standardized normal distribution. Equation (14) says that the price of an option on a stock without cash dividends depends on only five directly observable variables:

- the stock's price (S)
- the exercise price (E)
- the time to maturity (τ)
- the risk-free rate of interest (r_f)
- the volatility of the stock (σ)

Furthermore, it can be shown that

$$\frac{\partial C}{\partial S} > 0, \frac{\partial C}{\partial E} < 0, \frac{\partial C}{\partial \tau} > 0, \frac{\partial C}{\partial r} > 0, \frac{\partial C}{\partial \sigma} > 0. \tag{15}$$

3. Use GAs to Solve OPM

Assuming that an asset price S follows a stochastic process with $v(S)S^2$ denoting the *diffusion term* and rS the *risk-adjusting drift*, the partial differential equation characterizing all the contingent claims defined on the asset price is

$$L(C(S,\tau)) = \frac{1}{2}v(S)S^2\frac{\partial^2 C}{\partial S^2} + rS\frac{\partial C}{\partial S} + \frac{\partial C}{\partial \tau} \quad (16)$$

with the boundary conditions

$$C(S,0) = max(S-E,0), \quad C(0,t) = 0. \quad (17)$$

By Equation (11), to satisfy the no-arbitrage condition, the price of a European call, is given by

$$L(C(S,\tau)) - rC(S,\tau) = 0, \quad (18)$$

where $\tau = T - t$ is the time to expiration of the call.

The call price C can be approximated by C_a,[3]

$$C_a(S,\tau) = C_0(S,\tau) + \sum_{i=1}^{N}\psi_i(\tau)\phi_i(S), \quad (19)$$

where

$$C_0(S,\tau) = \begin{cases} S - Ee^{-r\tau}, & \text{if } S > E \\ 0, & \text{if } S < E \end{cases}$$

Or, alternatively, C_a can be written as follows.

$$C_a(S,\tau) = S - Ee^{-r\tau} + \sum_{i=1}^{N}\psi_i(\tau)\phi_i(S), \quad (20)$$

if the call option is *in-the-money* ($S > E$) and

$$C_a(S,\tau) = \sum_{i=1}^{N}\psi_i(\tau)\phi_i(S) \quad (21)$$

if the call option is *out-of-the-money* ($S < E$), where ψ_i and ϕ_i, i=1,...,N, are known analytic functions and are called *trial functions*, $C_0(S,\tau)$ is a function chosen properly to satisfy the boundary and initial conditions.

Notice that the choice of $C_0(S,\tau)$ here is different from the one used in Chen and Lee (1997). There, they did not disntiguish the case *in-the-money* and *out-of-the-money*. The failure to distinguish these two cases may be responsible for the overestimation of the call price in the *out-of-the-money* case.[4] Furthermore, if we only consider the in-the-money case for $C_0(S,\tau)$, i.e., $C_0(S,\tau) = S - Ee^{-r\tau}$ and suppose $\psi_i(0) = 0$, then in the *out-of-the-money* case $C_a(S,0)$ is $S - E$ rather than 0. Therefore, the boundary condition (17) is not satisfied.

The trial functions chosen to approximate $C(S,\tau)$ in this paper remain unchanged, i.e.,

$$\psi_i(\tau) = a_i\tau \quad (22)$$

and

$$\phi_i(S) = \frac{1}{1+e^{3.5iS}} \quad (23)$$

In addition to the boundary and initial conditions, it is desirable to have C_a which can also satisfy the signs of the five partial derivatives in Equation (15). Among them, the most important one is $\frac{\partial C}{\partial S} > 0$. $\frac{\partial C}{\partial S}$ is called the *Black-Scholes delta* or *hedge ratio*. It tells us how the call price will change in response to the change in the stock price. In the Black-Scholes model, the hedge ratio is $N(d1)$, which is between 0 and 1.

Given the choices of Equations (21) and (22), $\frac{\partial C}{\partial S}$ implies the following restriction,

$$\frac{\partial C_a}{\partial S} = 1 + \sum_{i=1}^{N}\psi_i(\tau)\frac{-3.5iSe^{3.5iS}}{1+e^{3.5iS}} > 0 \quad (24)$$

Based on the no-arbitrage condition, i.e.,

$$L(C(S,\tau)) - rC(S,\tau) = 0, \quad (25)$$

we shall define the error of our approximation R in terms of the linear operator L,

$$R = L(C_a(S,\tau)) - rC_a(S,\tau) \quad (26)$$

By the chosen trial functions, R can be derived analytically as follows.

$$\begin{aligned} R = & \sum_{i=1}^{N}\frac{\partial\psi_i(\tau)}{\partial\tau}\phi_i(S) + \sum_{i=1}^{N}\psi_i(\tau)[(1/2)v(S)S^2 \cdot \\ & \sum_{i=1}^{N}\frac{\partial^2\phi(S)}{\partial S^2} + rS\sum_{i=1}^{N}\frac{\partial\phi_i(S)}{\partial S}] \\ & - r\sum_{i=1}^{N}\psi_i(\tau)\phi_i(S) + 2rEe^{-r\tau} \\ = & \sum_{i=1}^{N}a_i\frac{1}{1+e^{3.5iS}} + (1/2)v(S)S^2 \cdot \end{aligned}$$

[3] Here, the *weight residuals method* extensively used in the numerical partial differential equation is applied. For reference, see Barucci et al. (1995).

[4] See Figures 3 and 4 of Chen and Lee (1997).

$$
\begin{aligned}
&|\sum_{i=1}^{N} a_i \tau \frac{(3.5i)^2 e^{3.5iS}(1+e^{3.5iS})^2}{(1+e^{3.5iS})^4} + \\
&\sum_{i=1}^{N} a_i \tau \frac{-2(3.5i)^2(1+e^{3.5iS})(e^{3.5i})^2}{(1+e^{3.5iS})^4}| + \\
&rS \sum_{i=1}^{N} a_i \tau \frac{-3.5ie^{3.5iS}}{(1+e^{3.5iS})^2} - r\sum_{i=1}^{N} a_i \tau \frac{1}{1+e^{3.5iS}} \\
&+ 2rEe^{-r\tau}
\end{aligned} \tag{27}
$$

In the next section, genetic algorithms are applied to the search for $\{a_i\}_{i=1}^{N}$.

4 Simulation Description and Results

Table 1: The Setting of Controlling Parameters

Number of chromosome	25000
Population size	50
Length of string	15
Selection mechanism	roulette-wheel selection
Crossover style	two–point crossover
Crossover rate	0.6
Mutation rate	0.001
Interval of parameters	[-5,5]
Fitness function (1)	$\sum_{S=0}^{5} R_S^2$
Fitness function (2)	$\sum_{S=0}^{5}(C_{a,S} - C_{BS,S})^2$

The interval of parameters $\{a_i\}_{i=1}^{N}$ is set to satisfy the condition $\frac{\partial C_a}{\partial S} > 0$ given that the interval of stock price is set to be [0,5] (See Table 2)

The software used in this paper is *GENESIS 5.0*, written by John Grefenstette (Grefenstette, 1990) to promote the study of genetic algorithms for function optimization.

Like Chen and Lee (1997), two fitness functions are considered in this study. The first one is based on the residuals defined by the Black-Scholes partial differential equation under different stock prices, i.e., the one defined in Equation (26). We shall denote these residuals by R_S where S are stock prices. The second one is simply based on the residuals defined by the difference between the approximating price C_a and the true price (the Black-Scholes price) C_{BS}. The second one is also frequently used in the application of **ANNs** to option pricing. The difference between these two measurements is that to have the former one, we must know the true model, e.g., the Black-Scholes model, while the latter does not require this knowledge. Therefore, by taking both fitness functions into account, we can evaluate the pricing performance of GAs not only for the case when the true model is known but also for the case when it is unknown. Given these two defined residuals, our chosen fitness functions are simply the sum of squared errors (**SSE**), namely, $\sum_S R_S^2$ and $\sum_S (C_{a,S} - C_{BS,S})^2$ (Table 1).

Table 2: The Parameters of the European Call Option

Stock price (S)	[0, 5]
Exercise price (E)	1
Time to maturity (τ)	1
Risk-free rate of interest (r_f)	0.1
Volatility of stock (σ^2)	0.1

Table 3: Estimated Coefficients of the Trial Functions (out-of-the-money): Fitness Function 1

parameter	a_1	a_2	a_3	a_4	a_5
GAs (N=1)	-3.42				
GAs (N=3)	-1.95	5.00	-3.56		
GAs (N=5)	0.001	-0.002	0.01	0.002	-0.01

Table 4: Estimated Coefficients of the Trial Functions (out-of-the-money): Fitness Function 2

parameter	a_1	a_2	a_3	a_4	a_5
GAs (N=1)	-3.43				
GAs (N=3)	-1.95	5	-3.56		
GAs (N=5)	0.001	0.001	0	-0.001	-0.001

The test problem is the European call option with the five parameters described in Table 2. In this study, **GAs** are applied to approximate the continuous call price function $C_{BS}(S)$ given that the other four parameters are fixed. The domain of S is set to be [0,5]. This domain is also different from that in Chen and Lee (1997). In Chen and Lee (1997), the domain was restricted to [0.6,5] and the case of *deep out-of-of-money* [0,0.6) was excluded, while in this paper, this part in included. Representative points

$\{S_i\}_{i=1}^{l}$ are sampled from this domain in the following manner: $S_1 = 0.1, S_{i+1} - S_i = 0.1, S_{l=50} = 5, \forall i$. Given E, τ, r_f, σ, the *no-arbitrage* prices can be obtained directly from Equation (14) for each S_i ($i = 1, ..., 50$) and they are depicted as the solid line in Figures 1 and 2. The performance of genetic algorithms is tested with the number of trial functions increasing from 1 to 3 and then to 5. The C_as computed from the five trial functions with the fitness functions 1 and 2 are depicted as a dash line in Figures 1 and 2 respectively. The coefficients estimated from different numbers of trial functions with the fitness functions 1 and 2 are exhibited separately in Tables 3 and 4 for the *out-of-the-money* case and in Tables 5 and 6 for the *in-the-money* case.

Table 5: Estimated Coefficients of the Trial Functions (in-the-money): Fitness Function 1

parameter	a_1	a_2	a_3	a_4	a_5
GAs (N=1)	2.30				
GAs (N=3)	2.30	2.30	2.3026		
GAs (N=5)	2.30	2.30	2.30	2.30	2.3158

Table 6: Estimated Coefficients of the Trial Functions (in-the-money): Fitness Function 2

parameter	a_1	a_2	a_3	a_4	a_5
GAs (N=1)	2.8727				
GAs (N=3)	2.6167	2.30	2.30		
GAs (N=5)	2.6114	2.30	2.30	2.30	2.3026

Table 7: Fitness of the GA Option Pricing

N	$\sum_S R_S^2$	$\sum_S (C_{a,S} - C_{BS,S})^2$
GAs (N=1)	7.5×10^{-2}	7.6×10^{-2}
GAs (N=3)	6.5×10^{-2}	6.5×10^{-2}
GAs (N=5)	4.5×10^{-3}	4.75×10^{-3}

One of the distinctive feature of this paper is to take into account the *asymmetric effect* on $C_a(S,\tau)$ between the case *in-the-money* and the case *out-of-the-money*. In Chen and Lee (1997) and Trigueros (1997), this separation is neglected and the poor fitness of GAs and GP for the *out-of-the-money* scenario is well known. Hence, we consider an effective way to overcome this problem is to take advantage of this *domain-specific* knowledge and put them explicitly into the design of GA. As a matter of fact, comparing the estimated coefficients in Tables 3 and 4 and Tables 5 and 6, we can see the significant difference in the $C_a(S,\tau)$ between the cases *in-the-money* and *out-of-the-money*. Nevertheless, different fitness functions seems to have negligible effect on the estimated coefficients. This can be seen by comparing Tables 3 with 4 and comparing Tables 5 with 6.

The fitness performance is summarized in Table 7. When the number of trial functions increases from one to three and further to five, the **SSE** derived from both fitness functions drops continuously. In addition to the absolute error, a relative measure, the *absolute percentage error* (**APE**), is also taken into account. The **APE** is defined to be $\frac{|C_{a,S} - C_{BS,S}|}{|C_{BS,S}|}$. The **APE**s under the fitness functions 1 are depicted in Figure 3. It is clear that the **APE** distribution is still *asymmetric*. When the option price is *in-the-money* ($S > E$), the **APE** is almost nil, and when the option price is *out-of-the-money* ($S < E$), the **APE** is high up to 100% (Figure 3).

To evaluate the effectiveness of seperating out-of-the-money from in-the-moeny, Figure 3 in Chen and Lee (1997) is replicated in Figure 4 here. By comparing Figure 3 with Figure 4, we can see that this paper make an significant improvement over Chen and Lee (1997) in the *out-of-the-money* case, more precisely, the *deep out-of-the-money* case. Take $S = 0.6$ as an example, which means that *the stock price slumps into only a half of its original price on which the strike price (E) is based.* The **APE** of Chen and Lee (1997) in this case is high up to 243%, while it is only 100% in this paper. Moreover, if we extend the $S = 0.6$ further down to $S = 0.1$, we can see that the **APE** in Chen and Lee (1997) can increase exponentially, while it is quite stable and is around 100% in this paper. Therefore, it confirms us the belief that an improvement can be made if we take the asymmetric properties of the call price function explicitly into account.

5 Concluding Remarks

In this paper, we apply genetic algorithms to option pricing by separating the *out-of-the-money* case from the *in-the-money* case. Some preliminary results on the improvement in terms of the **APE** are observed in the *deep out-of-the-money* case. However, the question which has not been addressed seriously is where the cutoff point is. Clearly, the answer is definitely not always *one*. In fact, we suspect that before the expiration date, the cutoff point should be *less than* one. A rigorous study to confirm this is left for the further studies.

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Figure 1:Call Price [Fitness Function 1,N=5]

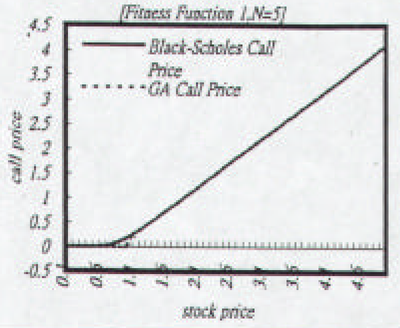

Figure 2:Call Price [Fitness Function 2,N=5]

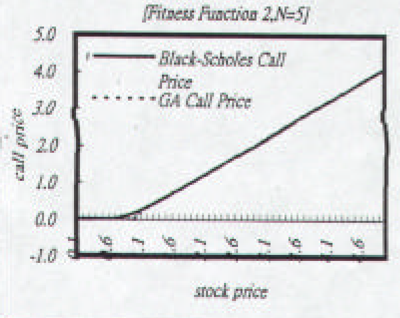

Figure 3:Absolute Percentage Error of Call Price[Fitness Function 1,N=5]

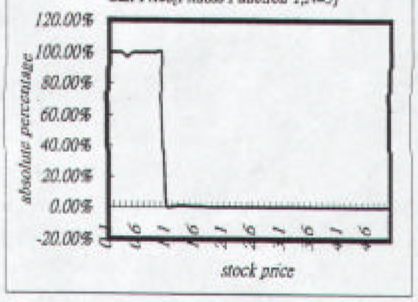

Figure 4:Absolute Percentage Error of Call Price [Chen and Lee (1997),Figure 3]